A CONCISE TREATISE ON

REINFORCED CONCRETE

A COMPANION TO
"THE REINFORCED CONCRETE MANUAL"

BY

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PREFACE

THE employment of reinforced concrete in engineering and architectural structures has rapidly become a common practice, and as the natural outcome of its more extended use descriptive articles in the technical press have brought the methods of construction to the prominent notice of professional readers. Numerous papers on reinforced concrete and kindred subjects have been read before technical institutions and societies, out of which interesting discussions have arisen, and these papers and discussions have appeared both in the special literature of the various bodies and in the columns of the technical press. As a consequence of the publicity given to the use of this material it has become possible to treat the subject in a considerably more condensed manner than was justified a few years ago, since details of construction and lengthy descriptions of experiments have no longer the special interest that they possessed in the immediate past. Although there are doubtless many engineers and architects who will still need the fuller treatment given in the larger work which the author has published conjointly with Mr. W. Dunn, still, it is evident that there is a great and increasing demand for a concise and handy volume, and it is hoped that this demand may be satisfactorily met by the present treatise.

During the winter of 1908–9 the author gave a series of lectures on reinforced concrete at the City and Guilds of

London Technical College, and during and after the delivery of these lectures many suggestions were made to the author that they should be embodied in book form.

The subject-matter of the lectures has been very considerably enlarged upon in writing the present volume, as it was impossible to deal with the subject except in a very sketchy manner in a series of six lectures illustrated with lantern slides, and it is hoped that, in conjunction with the "Reinforced Concrete Manual," this book may form a sufficiently complete treatment of the subject for all essential purposes.

The portions of the subject dealt with in the "Manual" have been omitted from the present volume or only briefly touched upon, but the reasoning on which the calculations for the design of structural members are based and the derivation of the fundamental equations have been fully explained. Lengthy descriptions of systems of construction and of experiments and tests have been omitted, but the results of tests and the conclusions to be drawn from them have been given full consideration in discussing the data on which the calculations are based, while a short description of the methods of reinforcement which are essential to good design has been given in the concluding chapter.

CHARLES F. MARSH.

June, 1909.

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REINFORCED CONCRETE

CHAPTER I

PROPERTIES

THE insertion of metal to strengthen brickwork and concrete is by no means a new thing, Wilkinson's patent for a suspended strap reinforcement for floors having been taken out in 1854, but within recent years the subject of reinforcing concrete with metal sections has received a good deal of attention, and under various names reinforced concrete has been brought forward rather prominently as a new material that may be used in suitable cases in place of or in conjunction with masonry or ironwork or both.

In the zest for the cause some of the advocates of reinforced concrete, it is to be feared, may have claimed somewhat too much for it. Like all other useful things, it has its limitations, yet from some of the publications one reads and statements one hears it might be thought that it was the cheapest, strongest, and best material to use in almost any engineering or architectural work which the art of man could design.

Although there are undoubtedly limitations to its use, the material is nevertheless an extremely useful one, and splendid results have been and will be accomplished by its employment.

Ten years ago there were few, in this country at any rate, who realised what was being done with reinforced concrete, and amongst those few it is certain that the majority looked upon the structures erected in this material with considerable distrust.

The structures undoubtedly existed, that fact could not be disputed, but that such lightness could be combined with strength was considered very doubtful by most of those who saw them.

This opinion is still held by many able engineers and architects, at any rate in regard to certain classes of structures, more especially those which are employed for the conveyance or the retention of water. The result which is most generally feared is the corrosion of the embedded steel and consequent gradual failure of the structure.

Ten or twelve years ago, and probably up to a more recent date, many of the constructors in reinforced concrete knew little or nothing of the true behaviour of the concrete and iron with which they were erecting architectural and engineering works, and when they did use any formulæ at all they were satisfied with those of a most empirical kind.

It must be presumed that the clients of these firms placed great trust in the specialist's practical engineering sense; as the science, such as it was, which was then used in the design of reinforced concrete was mainly regarded as the trade secret of the several patentees of the systems of reinforced concrete then in use on the Continent and in America. Such a state of things could not exist for long, and engineeers, particularly engineering professors, began to inquire into the nature of the stresses set up in a dual material formed of two such very different substances as concrete and iron.

Even in the early years of the present century the most scientific formulæ used by many of the constructors in reinforced concrete were semi-empirical, and those who attempted to design their structures by the application of true principles had to face the difficulty of competition with firms whose methods were less accurate.

One of the chief disadvantages pertaining to the use of reinforced concrete, even to-day, is the time and labour spent upon the calculations, if these are made in a scientific manner, and the consequent expense of design. Even with the use of the many tables and diagrams which have been published, and which are based on proper and efficient formulæ, there is still considerable labour in the design (specially when the structure is a complicated one), if the calculations are to be carried through in a scientific manner from start to finish.

In what may be called the early days of reinforced concrete, few, if any, experiments had been carried out on this material, except such rough tests as were made by the several constructors mainly for the purposes of advertisement, and no accurate measurements had been made to discover the behaviour of the materials under service conditions, and as a consequence of the absence of scientific investigation there was little available data on which to base a proper system of calculation.

Within recent years innumerable experiments on small-sized specimens, and also many tests on full-sized pieces, have been carried out with instruments for the exact measurement of the strains produced under various conditions of loading. We have thus obtained a scientific knowledge of reinforced concrete on which satisfactory theories may be based.

When we come to inquire more closely into the subject we find that, owing to the conditions existing in practice and to the fact that large factors of safety must be allowed, even in the most scientific design, sufficient accuracy can be obtained, although some departure may be made from the absolute scientific facts which have been derived from the results of carefully conducted experiments.

When such departures are made from what we believe to be the true theory, it is very necessary that it should be known what such departures involve and what are the limitations to the use of any empirical or semi-empirical formulæ arrived at by their adoption. In the same way, although most excellent and useful diagrams and tables have been published to simplify the design of such pieces as beams, slabs, columns, etc., it is very advisable that they should not be used without a knowledge as to the methods by which they were calculated. For these reasons it is most essential that all architects and engineers should, in the future, have a knowledge of the general principles of the theory necessary for the proper design of simple reinforced concrete structures. From such knowledge they may readily proceed more deeply into the study of this interesting material and to its applications in more complicated structures if their professional practice so requires.

With regard to the practical side of reinforced concrete, this is perhaps rather more the business of the builder or contractor than of the architect or engineer, but every designer should thoroughly understand the difficulties which have to be met in the erection of the structure he is designing, and no good specification can be drafted without an intimate knowledge of the materials to be used and the conditions to be dealt with in the construction of the work.

The study of the practical side of reinforced concrete construction is very necessary, since the intelligent appreciation of the methods which may be adopted by the builder or contractor in the erection of any structure which one may have to design will often reduce the cost of the work in a very considerable degree.

The theoretical side can be learnt to a large extent in the lecture room, but, although the practical side may be dealt with and some information given, still there is no getting away from the fact that the seeing of works in progress of construction and experience gained on such works are the only ways of obtaining full appreciation of the methods adopted in the execution of the details.

The properties of reinforced concrete which have brought it to the fore as a building material are mainly resistance to fire and heat, durability, resistance to shocks, capacity for moulding, facility of connecting together of parts, speed of erection and economy of cost. The properties which are disadvantageous to its use are its hardness and impenetrability, transmission of sound, and the necessity of care during construction.

Fire Resistance.

The fire-resisting qualities of reinforced concrete are now fully appreciated, but the tests which have been carried out and the actual fires which have occurred during the last few years have clearly demonstrated the necessity of giving sufficient cover of concrete to the reinforcements and the advisability of the careful selection of aggregate when the highest degree of fire resistance is required.

As to the covering of concrete over the metal it is evident that more thickness is required for narrow surfaces, such as beams and columns, where the flames can wrap round the member, than for wide flat surfaces where no such envelopment can take place. This is very noticeable in tests and the appearance of buildings after actual fires have occurred.

It has been found that according to the degree of fire resistance required for beams and columns, the protective thickness of concrete should be from $1\frac{1}{2}$ inches to 2 inches and all the arises should be chamfered or rounded for 2 to 3 inches. The covering over the bars of flat slabs may be reduced from $\frac{3}{4}$ of an inch to 1 inch.

Some special tests were carried out by the British Fire Prevention Committee in October, 1905, on the resistance of several of the aggregates commonly employed, when subjected to intense heat and subsequent application of a jet of water under high pressure. It was found that furnace slag, coke breeze, broken brick, furnace clinker, and burnt ballast were best. Then followed broken granite while the Thames ballast slab gave the worst result, deflecting and cracking badly and suffering much disintegration on the application of the water jet on the hot surface.

Coke breeze, furnace clinker, and similar substances make weak concretes, and are consequently unsuitable for structures bearing heavy loads; they are undoubtedly the best materials for concrete when used for fire protection only and not as a load-bearing part of the structure, if proper care is taken in their selection to eliminate under-burnt particles.

The Committee appointed by the Royal Institute of British Architects to formulate recommendations for reinforced concrete works made the following observations when dealing with fire resistance:—

"(1) Floors, walls, and other constructions in steel and concrete formed of incombustible materials prevent the spread of fire in varying degrees according to the composition of the concrete, the thickness of the parts, and the amount of cover given to the metal.

"(2) Experiment and actual experience of fires show that concrete in which limestone is used for the aggregate is disintegrated, crumbles and loses coherence when subjected to very fierce fires, and that concretes of gravel or sandstones also suffer, but in a rather less degree (the smaller the aggregate the less will be the injury). The metal reinforcement in such cases generally retains the mass in position, but the strength of the part is so much diminished that it must be renewed.

"Concrete in which coke breeze, cinders or slag form the aggregate is only superficially injured, does not lose its strength, and in general may be repaired. Concrete of broken brick suffers more than cinder concrete and less than gravel or stone concrete.

"(3) The material to be used in any given case should be governed by the amount of fire resistance required as well as by the cheapness of, or the facility of procuring, the aggregate.

"(4) Rigidly attached web members, loose stirrups, bent-up rods, or similar means of connecting the metal in the lower or tension sides of beams or floor slabs (which sides suffer most injury in case of fire) with the upper or compression sides of beams or slabs not usually injured, are very desirable.

"(5) For main beams a covering of $1\frac{1}{2}$ inches to 2 inches of concrete over the metal reinforcement appears from experience in actual fires to afford ample protection to the structural parts. In floor slabs the cover required may be reduced to 1 inch.

"All angles should be rounded or splayed to prevent spalling off under heat.

"(6) More perfect protection to the structure is required under very high temperature, and in the most severe conditions it is desirable to cover the concrete structure with fire-resisting plastering which may be easily renewed.

"Columns may be covered with coke-breeze concrete, terra cotta, or other fire-resisting facing."

Conduction of Heat.

Reinforced concrete is a poor conductor of heat, and consequently even temperatures should be easily maintained in rooms having reinforced concrete walls. These walls are, however, much thinner than brick or stone, and it has been stated that the temperature of such rooms varies considerably with that outside. It is inadvisable to speak definitely without actual experience, but it would certainly appear that a $4\frac{1}{2}$ inch reinforced concrete wall will keep a room at a very much more even temperature than will a $4\frac{1}{2}$ inch brick wall.

Of course the use of double walls will ensure that the rooms will always have an even temperature.

DURABILITY.

No very definite opinion can be given as to the durability of reinforced concrete under various conditions for some time to come, but for floors and similar structures, where the load is not subject to great and sudden variations and there is no fear of dampness penetrating, even the least sanguine must acknowledge that its life will be considerable.

As to the life of structures where water may penetrate the mass, where the loading is variable or where subjected to vibrations, there may be more doubt, but if properly designed, constructed, and protected by waterproofing where necessary, there is every reason to suppose reinforced concrete in any of these situations will be as durable as if it was only lightly loaded and kept free from the effects of water.

Whatever the ultimate life may be, there is no doubt as to the very small cost for maintenance of a reinforced concrete structure, since no painting, pointing, or patching is necessary as is the case with other materials.

RESISTANCE TO SHOCKS.

Reinforced concrete structures show great resistance to shocks. In countries subjected to earthquakes this material will doubtless be employed to a very considerable extent for buildings. Reinforced concrete is being largely used in the rebuilding of San Francisco and Kingston, and it will no doubt be employed extensively when the rebuilding of Messina is taken in hand.

EASE AND RAPIDITY OF ERECTION.

The ease with which pieces may be moulded in concrete renders it a very suitable material for building, and by a proper selection of materials it is possible to reproduce any ornamental design by moulding concrete in a plaster of paris cast, but concrete is perhaps more suitable to bold mouldings, and it is almost certain that the architecture of concrete will develop along the line of expansive surfaces relieved by bold and heavy strings, cornices and other mouldings.

The metal reinforcements of the composite structure require no bolting or riveting for fastening them together. A little bending, lapping with wire or joining by loose or screwed sleeves is all that is required, the frictional resistance and adhesion between the concrete and steel does the rest.

Reinforced concrete buildings are frequently erected in an extraordinary short space of time, and this speed of construction is a great advantage, especially for factories and similar structures.

The time required for the concrete to properly set may be placed against this advantage, as although a structure may be quickly finished the loading of any part to anything like its calculated load must be avoided until the concrete has been allowed to set for two or three months.

Economy of Cost.

As regards the cost of construction it has been found that in cases where reinforced concrete lends itself to the office it has to perform, a saving of 15 to 20 per cent. can be effected by its use.

Of course this saving cannot be expected always, and in some instances where this form of construction has been used it has been found to be more expensive than other more suitable material; but owing to the thinness of concrete required and the simplicity of the construction, there is no doubt that it is a cheap form of construction if it is rationally employed.

The provision, erection and removal of the false work and moulds form a large percentage of the cost of any reinforced concrete structure, as the false work and moulds require a considerable amount of carefully prepared timber. This item in the cost can be controlled to a great extent by forethought and good management.

Hardness.

The great hardness and impenetrability of well-made concrete makes it necessary to provide for all fastenings, openings, etc., before moulding. The cutting of a chase in a reinforced concrete wall or floor, which has been moulded for a few months, is no light task, and nails cannot be driven into reinforced concrete unless coke breeze, cinder, or similar material has been used as the aggregate.

Conduction of Sound.

The thinness of walls necessary when reinforced concrete is used for buildings renders them very noisy—sound penetrates concrete in a marked degree, causing street noises, typewriting in an adjoining room, and similar nuisances, to be very noticeable. Double walls are of course a remedy for this, but they add considerably to the cost.

Partition walls, unless they have to carry a load or are in a situation where noise does not matter, need not be made of reinforced concrete, while outside walls should be made double, if the building is for office or domestic purposes and comfort is required.

Care in Construction.

The necessity of care during the building of structures of reinforced concrete is continually brought forward as an argument against its use, but it is certain that as reinforced concrete comes more and more into use it will take its place amongst other building materials, its manipulation will be as well understood and the necessity of care during construction as much recognised, as is the case with framed steel structures. There is no doubt that care and good supervision are necessary when building in reinforced concrete, but, as time goes on, a greater and greater number of workmen and foremen are being trained, and it certainly appears that the men, for the most part, take an interest in their work, recognising that something more rests on their carefulness when carrying out the necessary operations for the erection of a reinforced concrete structure, than when putting in ordinary concrete foundations.

When viewing buildings during construction it is frequently noticed that the workmen are not so careless as their superiors. One has seen buildings during construction where floors have had every appearance of being dangerously loaded with sand, broken stone, and other materials and plant, and where dimensions have the appearance of being cut rather finer than the best conservative design would admit. Such procedures are not calculated to act as good examples to the workmen or to induce them to use care in the proper placing of the materials. Such evils as these will, of course, be reduced when engineers and architects become better acquainted with the material, and embody more stringent clauses in their specifications. Although it is frequently convenient for a contractor to deposit materials and plant on floors of reinforced concrete, due care should be taken not to overload them, as, although a structure may to all appearance safely bear such loading, it is extremely difficult to tell what stresses may be set up within the material tending to lower its ultimate resistance.

At the present time, when reinforced concrete is forcing its way to notice, it is doubtless tempting to reduce the sizes of the members of a structure in order to show off the capabilities of the material, but, when this is done, that useful coefficient, "the factor of safety," is being encroached upon. Although, if all is well with the material, such a structure may resist the loading to which it is subjected, with every appearance of safety, and its users may never know how near they are sailing to the wind; it is absolutely immoral, while man remains human, to reduce that safeguard over bad material or workmanship and all other adversities and vicissitudes, which we call the factor of safety.

Watertightness.

There is still much difference of opinion as to the capability of concrete to resist the penetration of water, and this is rather a dangerous subject to be in any way dogmatic upon.

There is no doubt that reinforced concrete standpipes have been constructed which have withstood considerable heads of water, and it is equally certain that very many reinforced concrete reservoirs and tanks leak more or less.

For small heads, say up to 20 feet or so, concrete, when made of good materials carefully selected, and where the mortar fills the voids as nearly as is possible, will be water-tight, if it remains uncracked. When concrete has been used for small diameter pipes with comparatively thick shells considerable pressures, say, up to 100 feet head, have been resisted, but as pipes get larger in diameter there is more concrete to each foot length of pipe, and the more concrete the greater risk of local defects. With the use of coatings it is possible, doubtless, to construct a large diameter pipe, say up to 5 or 6 feet, to withstand a head of 60 or 70 feet, but there is always risk of local defects.

In any situation where the surface may be subjected to considerable changes of temperature or humidity, concrete is very liable to crack, and in such a case, however well it is able to resist the passage of water as a whole, the formation of cracks may render it absolutely useless in this respect.

Where no special precautions are taken with regard to waterproofing, it is advisable, if it is possible, to place the concrete of a structure in such a situation in dry and cold, but not frosty, weather; as temperature cracks are formed when the concrete contracts, while the effects of expansion will only tend to consolidate the mass.

A reinforcement near the surface exposed to changes in atmospheric conditions will prevent any dangerous cracking of the concrete due to local failure, and will distribute the contraction in such a manner that the cracks, although numerous, are individually imperceptible. Such reinforcement should always be used in a structure which has to resist the passage of water.

For unimportant cases a sufficient degree of impermeability will be attained, if good, well-mixed materials are used for the concrete and about 5 per cent. to the weight of the cement, of finely divided slaked lime is added when mixing the ingredients, the structure being also reinforced by a light mesh reinforcement near the exposed surface. For important works, however, there seems no doubt that a special layer of waterproofing, protected by an outer covering of concrete or other material, should be inserted near the surface exposed to water pressure.

There are many excellent sheetings prepared with bitumen which may be used for this purpose, but if the best results are desired it is advisable to use at least two layers, the laps of the one layer breaking joint with those of the other. The ordinary sheetings prepared on burlap are somewhat expensive when used in this way, but systems have recently been introduced in which thin layers of felt are used, breaking joint with each other, and having a bituminous mastic sandwiched between them.

Preservation of Steel.

The question of the preservation of the steel in reinforced concrete has been frequently discussed, and it has been shown again and again that, when the work is properly executed with good materials, there is absolutely no fear that the steel will rust. In spite of most emphatic assurances and conclusive proofs, doubt still continues to exist on this point.

Very severe tests have recently been carried out in

Germany on thirty-two beams, which were subjected to rapid rusting while being loaded. These tests showed that, provided the reinforcement was not stressed beyond its elastic limit, the cracks formed in the concrete were insufficient to induce rusting.

The concrete was mixed in the proportions of 1 : 2 : 4, and the reinforcement was from 1·03 to 1·31 per cent. of the sectional area above its axis, which was placed 1¼ inches above the bottom of the beams.

A sheet-iron casing was placed around the centre third of the beams, and a mixture of carbonic acid gas, oxygen, and steam was passed through this casing.

A mixture of carbonic acid gas and water was found to have no effect on unprotected steel bars, but with the addition of oxygen the rusting was very rapid.

The beams were generally kept in this rusting mixture of carbonic acid gas, steam and oxygen for three working days, the loading being kept on during this period. After this treatment the concrete was cut away so as to expose the bars. In twenty-seven out of the thirty-two beams no rusting was found, although the loading must have stressed the steel to from 18,000 to 35,000 lbs. per square inch.

In the other beams, in which the steel was stressed to from 35,000 to 40,000 lbs. per square inch, the bars were found to be more or less rusted.

Some experiments on the rusting of steel in concrete have recently been carried out for Sir John Brunner at the National Physical Laboratory. Five specimens were tested, each being in the form of a prism of concrete, 7½ inches by 7½ inches sectional area, containing one turned bar of

1 inch diameter, and one $1\frac{1}{2}$ inches square bar from which the shop scale was not removed.

The blocks were covered with water several times a week for a year, and for a further three months were left in the open under the influence of the weather. One of the blocks was then broken up, and no trace of any action on the steel could be detected, even under microscopic examination.

Many instances have been recorded, showing that the steel in reinforced concrete is very adequately protected against rusting. And it is obvious that any slight amount of moisture which may get into the concrete, even if it reached the steel, could not cause rusting without an ample supply of oxygen. It is well known that steel, if kept under water, will rust very slowly, while, if it is exposed to the action of the atmosphere it oxidizes with great rapidity.

It is also well known that one of the best preservatives for steel work is a coating of cement grout.

A slight coating of rust on the bars, when they are embedded, is not in any way detrimental, providing all loose scale is removed, as the cement appears to enter into chemical combination with the oxide of iron, and a bar embedded in concrete in a rusted condition will be found to be quite free from rust if the concrete is broken away after some time has been allowed to elapse.

CHAPTER II

BEHAVIOUR UNDER LOADING

Compressive Resistance of Concrete

A very great number of tests have been made to determine the compressive resistance of concrete. It is not proposed to deal with these in any detail, but only to draw general conclusions from their results in so far as they effect the design of reinforced concrete structures.

The strength of concrete appears to vary approximately with the proportion of cement to the sand in the mortar employed, provided that the quantity of stone is such that every piece is completely surrounded with mortar. No additional strength is obtained by reducing the amount of stone beyond this quantity; in fact, the strength is frequently decreased by so doing, the mortar when tested by itself being weaker than a dense concrete made with it.

If the voids in the stone are measured, sufficient mortar should be used to equal the capacity of the voids plus about 5 per cent.

Concrete, mixed in the proportions of 1 : 2 : 4, or 630 lbs. of cement to 13½ cubic feet of sand and 27 cubic feet of stone, the mixture usually employed for reinforced concrete structures, will have a strength of about 2,400 to 2,500 lbs. per square inch when one month old.

Other concretes made with mortars having different compositions, will have strengths varying approximately with the proportion of cement in the mortar.

Consequently, the strengths of concretes mixed with mortars of the following compositions may be assumed to have the strengths given in the fourth column of Table I. when one month old :—

TABLE I.

Mortar.	Proportion of cement in the mortar.	Proportion of cement in the mortar relative to 1 : 2 mortar.	Crushing strength, lbs. per square inch.
1 : 1	0·5	1·50	3,600
1 : 1½	0·4	1·21	2,900
1 : 2	0·33	1·00	2,400
1 : 2½	0·286	0·867	2,080
1 : 3	0·25	0·75	1,800

The resistances of concrete of coke breeze or furnace ashes is about one-third that for concrete of broken stone or shingle, and may be even less.

The strength of concrete increases with the age, the increase in the strength being rapid up to an age of about four months, when it will be about 30 per cent. greater than at one month, the strength after this period increases at a slower rate up to three or four years, when it will be about 50 per cent. greater than at the age of one month.

This is a very valuable quality for a structural material to possess, as the factor of safety increases with the age of the member, and it is this quality that the more rash of the firms constructing in this material mainly rely upon when they use a high working stress in designing structures.

The strength of concrete is effected to a considerable extent by the amount of water used in mixing. The strongest mixture is that which looks like moist earth when mixed, and in which a slight moisture appears on the surface after severe ramming, but a moderately wet mixture which will quake when rammed is very little weaker and much safer to use.

Like all other materials, concrete offers greater resistance in compression when subjected to flexure than when under direct compression; this is probably due to the aid given to each other by contiguous fibres caused by the shearing resistance of the material when the member is bent. For this reason it is the universal practice to allow higher safe resistance in compression for pieces under flexure than for those under direct compression. Allowing a factor of safety on the ultimate resistance of about four for pieces under flexure and about five for pieces under direct compression, will give for 1 : 2 : 4 concrete, the safe compressive resistances of 600 and 500 lbs. per square inch under flexure and direct compression respectively.

Professor Talbot, of the University of Illinois, has found from experiments on concrete columns that the stress deformation curve under compression has a distinct bend at a deformation of about half the ultimate; this deformation occurs at about three-quarters the ultimate resistance, and from this he very rightly considers that three-quarters the ultimate rather than the ultimate strength should be taken as a basis from which to assess the safe working stresses. On the basis of three-quarters the ultimate resistance of 1 : 2 : 4 concrete, the factors of safety for the values of 600 and 500 lbs. per square inch would be about 3 and 3⅗ respectively.

Elastic Behaviour under Compression.

To properly understand the derivation of the necessary formulæ for the design of reinforced concrete structures it is essential that the behaviour of the material under loading should be properly understood.

A great number of experiments have now been carried out on beams, columns, etc., and from these it is possible to obtain a knowledge of the deformations in a stressed member.

If a prism of any material is subjected to compressive or tensile stress it contracts or extends in the direction of the load, and at the same time expands or contracts laterally. With materials such as steel, if the loading does not stress the material above its elastic limit, the prism will resume its original dimensions upon the removal of the load. When, however, the loading exceeds the elastic limit of the material, it will not resume its original dimensions, but becomes permanently deformed or takes a permanent set.

Concrete has a very low elastic limit, in fact it may be said that for practical purposes its elastic limit is *nil.* It therefore takes a permanent set under very small loads.

When we are dealing with a material such as reinforced concrete, which is in reality two materials acting together in resisting the stresses, it is essential that the elastic properties of the two materials should be known.

A material is said to be truly elastic so long as its elastic limit is not exceeded.

During the truly elastic phase the deformation is directly proportional to the applied stress. This means, for instance, that if under an intensity of stress of 10,000 lbs. per square inch a bar of steel elongates or contracts 0·00033 of its length, then for a stress intensity of 20,000 lbs. per square inch it will elongate or contract 0·00066 of its length. In other words, the strain is proportional to the stress. This is what is known as Hooke's Law. There is still, in spite of the advance of technical training, a considerable confusion between the terms "stress" and "strain." The stress is the force acting upon a body and the strain is the deformation due to that force.

By common use it is customary to employ the term "stress" as meaning the intensity of stress. For instance, we talk of a stress of so many pounds to the square inch. Similarly, with strain, the strain in a bar of steel 20 inches long under a load of 10,000 lbs. per square inch would be 0·0066 inches, but we generally mean by strain the deformation per unit length, or in this case 0·00033. When stress and strain are referred to subsequently they will have these meanings.

The modulus of elasticity gives the relation between the stress and the strain for any material. It is the stress which would, if the material remained truly elastic, cause a strain of 1 or a deformation equal to the length of the specimen. In any given case, the modulus of elasticity multiplied by the strain, and divided by one, or the strain produced by a stress equal to the modulus of elasticity, equals the stress which produces the strain under consideration, or expressed mathematically, where E is the modulus of elasticity, f is the stress and λ = the strain.

$$E : 1 : : f : \lambda \text{ or } \frac{E\lambda}{1} = f. \qquad [1]$$

The way in which the modulus of elasticity is used for the purposes of calculating the resistances of reinforced concrete members may be explained briefly as follows:—

We know that (within the truly elastic phase) the stress is directly proportional to the strain. Now if a certain stress is applied to a fibre of a material it induces a strain, and this strain having been produced, no further deformation occurs, though the stress still acts on the fibre, or, in other words, the fibre is in equilibrium.

When equilibrium occurs in a body the action and reaction must be equal and opposite, consequently the straining of the fibre causes internal stress which is equal and opposite to the applied stress; this internal stress is called the resistance of the fibre. Now, since the resistance of the fibre is equal to the applied stress we have from the equation [1] given above

$$\lambda = \frac{f}{E} \quad . \quad . \quad . \quad . \qquad [2]$$

Now if two fibres of different materials are acting together to resist a stress, and under this stress they are equally deformed (either elongated or contracted), then it is evident (provided neither of them pass the truly elastic phase) that, as the strain is the same for both fibres, the ratio of the resistances of the fibres varies as their moduli of elasticity; or if f and c are the resistances of the two fibres and E_f and E_c are their coefficients of elasticity,

$$\text{then } \lambda = \frac{f}{E_f} = \frac{c}{E_c} \text{ or } \frac{f}{c} = \frac{E_f}{E_c}. \quad . \quad . \quad . \quad . \qquad [3]$$

It is most important that this is thoroughly understood, as the calculations for the design of reinforced concrete must be based on the elastic behaviour of the two materials.

In some materials the relation of the stress to the strain is different in tension than in compression, but with steel they are practically the same.

In the case of concrete, however, we are met with difficulties with regard to this relation.

In the first place it has no truly elastic phase or such a limited one as to be of no practical importance, and as a consequence it can only be said to have an instantaneous modulus of elasticity, or, in other words, its modulus of elasticity varies with the stress.

In the second place the elastic behaviour of concrete varies according to its composition and its age. Also, its modulus of elasticity in tension is much less than that in compression. This latter peculiarity, however, is not of great consequence, as the tensile resistance of the concrete is neglected in the formulæ for the design of reinforced concrete pieces under bending, the elastic behaviour of the concrete under tensile stresses being only of importance when the elongation of a member is required, which is seldom necessary in ordinary practice.

Then, further, by reason of its poor elastic qualities, concrete takes permanent sets under very small loads, and consequently if a compressive load is applied and then removed the concrete does not regain its original length.

When the same load is again applied and removed, it will be found to be shorter than before, but to a lesser degree, and so on for very many applications, the permanent set for each application becoming smaller and smaller, until at last, on the load being applied, it regains the length it had prior to this application of the load.

The diagram (Fig. 1) shows this for a gradually applied and removed load. The small circles between the curves indicate the deformations under repeated loadings.

It will be noticed that the load deformation curve becomes flatter under each application of the load and that there is a permanent set left after each loading, which

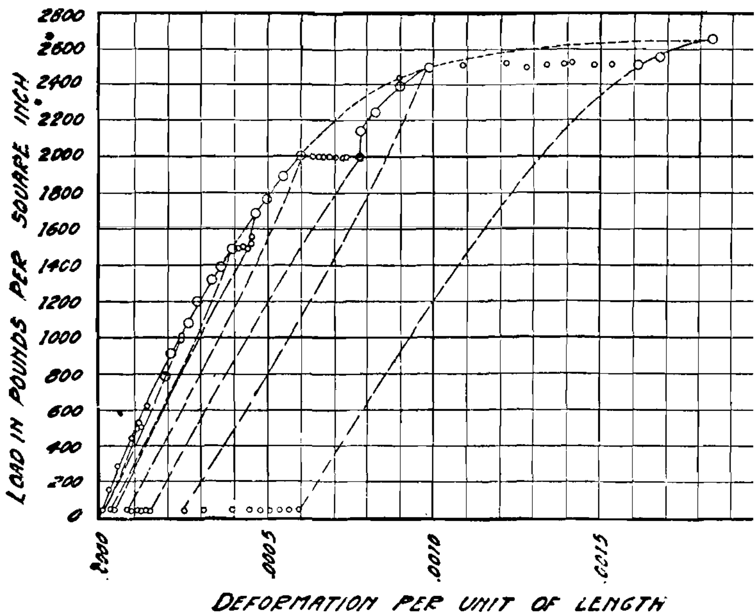


FIG. 1.

becomes less and less as the number of loadings increases. It appears that the deformation of the member increases at each repetition of the load in a greater degree than the permanent set. As will be seen, this behaviour of concrete affects its modulus of elasticity, since, after each loading the strain becomes greater, and as $\frac{\text{stress}}{\text{strain}} =$ modulus of elasticity, the modulus of elasticity decreases with the number of loadings.

For the reasons given, it is evident that the modulus of elasticity for concrete is a very doubtful quantity; fortunately, however, the variation of this factor has no marked effect on the calculations, for instance in the calculation of a member under direct compression, with 5 per cent. of reinforcement (a very large amount), a variation of 50 per cent. in the value of the modulus of elasticity of the concrete only alters the resistance of the piece by about 17 per cent.; for 1 per cent. of reinforcement the same variation of the modulus of elasticity will only alter the resistance about 3½ per cent.

The many experiments which have been carried out on compression pieces enable us to arrive at a fairly accurate value for the modulus of elasticity of concrete mixed in the proportions of about 1 : 2 : 4, the mixture which is generally adopted for reinforced concrete at the present day. The calculations will not be greatly affected if the same value is adopted for the somewhat richer mixtures which are sometimes employed. In any case no great accuracy can be obtained, as so many influences occur which may affect this coefficient.

Some authorities advocate the use of the initial modulus of elasticity or that given by the tangent to the stress-strain curve for concrete at its commencement. Professor Talbot, from his experiments carried out in 1906, found the initial modulus to be about 2,350,000, and from tests carried out at the Watertown Arsenal he obtained a value of about 2,500,000. From some later tests in 1907, he obtained an average value of 3,040,000, and explained the variation as being caused by the employment of a denser concrete.

Professor Talbot advocates the adoption of a parabolic stress-strain curve for concrete for the purposes of calculation, and with such an assumption it is necessary to use the initial modulus of elasticity, but for the straight line stress-strain curve usually assumed the modulus of elasticity used in the calculations should have a value lower than the initial modulus.

When a straight line stress-strain curve is assumed it appears better to take, for the modulus of elasticity, the value given by the tangent to the stress-strain curve of concrete at the working load which is to be allowed or at a stress of about 600 lbs. per square inch, when it will be considerably less than the values given by Professor Talbot, and more nearly 2,000,000.

When inquiring into the behaviour of concrete for the purposes of obtaining the modulus of elasticity for use in the design of reinforced concrete it is advisable to study its behaviour when combined with the steel.

If a compression member of reinforced concrete is tested under gradually increasing loads and the results plotted on a diagram, we get a curve as shown (Fig. 2); this is the elastic curve of the whole piece. Now the steel is perfectly elastic, and consequently we can find how much of the load it resists by multiplying its unit deformation at varying loads by its modulus of elasticity and by its area. The remainder of the total load, at each loading, must be borne by the concrete.

We can therefore find the true stress on the concrete alone by dividing the load thus found as being resisted by the concrete by the area of the concrete in the specimen.

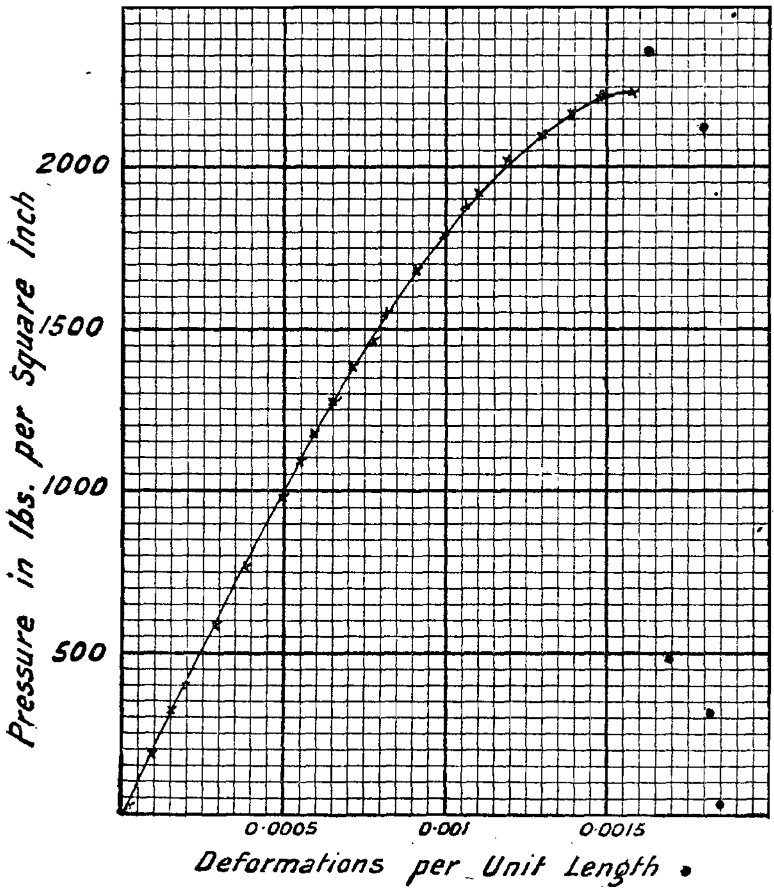


Fig. 2.

The new values for the stress plotted as ordinates in the same way as on the diagram (Fig. 2), with the original deformations as abscissæ, give a new curve which is the elastic or stress-strain curve of the concrete only.

Professor Talbot effects this graphically in the following manner: After having plotted the load-deformation or stress-strain curve for the whole piece, he finds the unit resistance of the steel for one loading, as described above, but instead of multiplying this by its area to obtain the total load resisted he multiplies it by the ratio of steel to the concrete in the section, which gives the load resisted by the steel per unit area of the entire section; he then plots this on the diagram and draws a straight line through the point so found, which gives the line of the stresses resisted by the steel per square unit of entire section. The vertical distance between this line and the load-deformation curve gives the stresses resisted by the concrete alone.

The steel line is straight, as the strain is always directly proportional to the stress.

Supposing the unit deformation or the strain under any load = 0·001 and the modulus of elasticity of steel is 30,000,000. The resistance of the steel is 0·001 × 30,000,000, or 30,000 lbs. per square inch.

Now suppose the prism had an area of 100 square inches and there was 1·2 per cent. of steel in the section, then the total load resisted by the steel would be 30,000 × 100 × 0·012, or 36,000 lbs. If this is deducted from the total load and the remainder divided by the area, we get the stress resisted by the concrete for the unit deformation or strain under consideration. These values must be replotted on the diagram to give points on the stress-strain curve for the concrete alone.

According to the method used by Professor Talbot, the

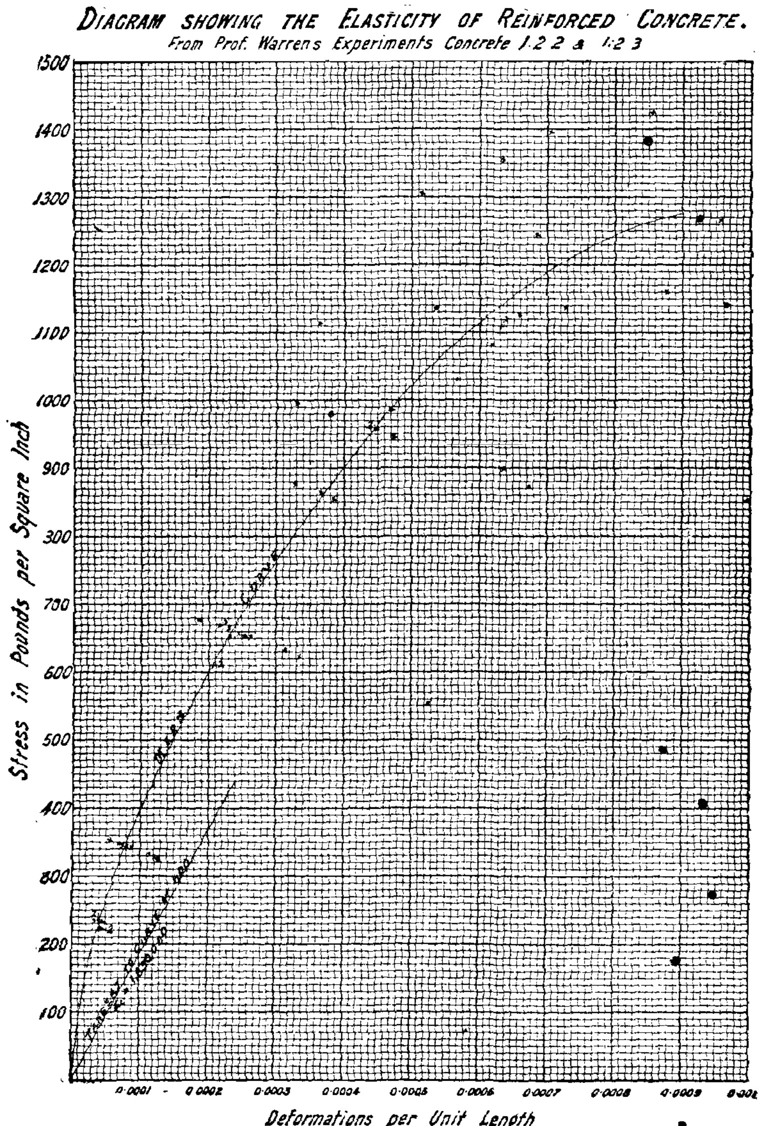


DIAGRAM SHOWING THE ELASTICITY OF REINFORCED CONCRETE.
From Prof. Warren's Experiments Concrete 1.2.2 & 1:2.3
Stress in Pounds per Square Inch
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
Mean Curve
0·0001
0·0002
0·0003
0·0004
0·0005
0·0006
0·0007
0·0008
0·0009
0·001
Deformations per Unit Length

FIG. 3.

stress on the steel would be found to be 30,000 lbs. per square inch as before, but this would only be multiplied by the ratio of the steel area to the concrete area, or 30,000 × 0·012 = 360.

This would be plotted as an ordinate for the unit deformation of 0·001, and a straight line drawn from the origin through the point so found. Now if the vertical distances between this line and the stress-strain curve for the reinforced concrete prism are taken off with dividers and set up from the horizontal axis at the unit deformations or strains at which the distances were measured, the points so found will give the stress-strain curve for the concrete alone.

The diagram (Fig. 3) shows the mean stress-strain curve for the concrete alone, drawn through points found from various experiments by the first method, the stresses resisted by the concrete being calculated directly for the several unit deformations. This curve gives 1,830,000 as the value for the instantaneous modulus at a stress of 600 lbs. per square inch.

Diagram (Fig. 4) shows the method adopted by Professor Talbot; the mean stress-strain curve being drawn from the results obtained from tests of reinforced concrete columns made by Professor Talbot in 1906 and the stress-strain curve for the concrete located from the steel line.

In this case the instantaneous modulus of elasticity for the concrete at a stress of 600 lbs. per square inch is 1,730,000.

In members of reinforced concrete subjected to direct compression and reinforced with longitudinal bars, where E and E_c are the moduli of elasticity of the steel and concrete respectively, the resistance which can be obtained from the reinforcements can only be $\frac{E_s}{E_c}$ times the resistance of the concrete as has been shown already.

It would appear, therefore, that the lower the value of

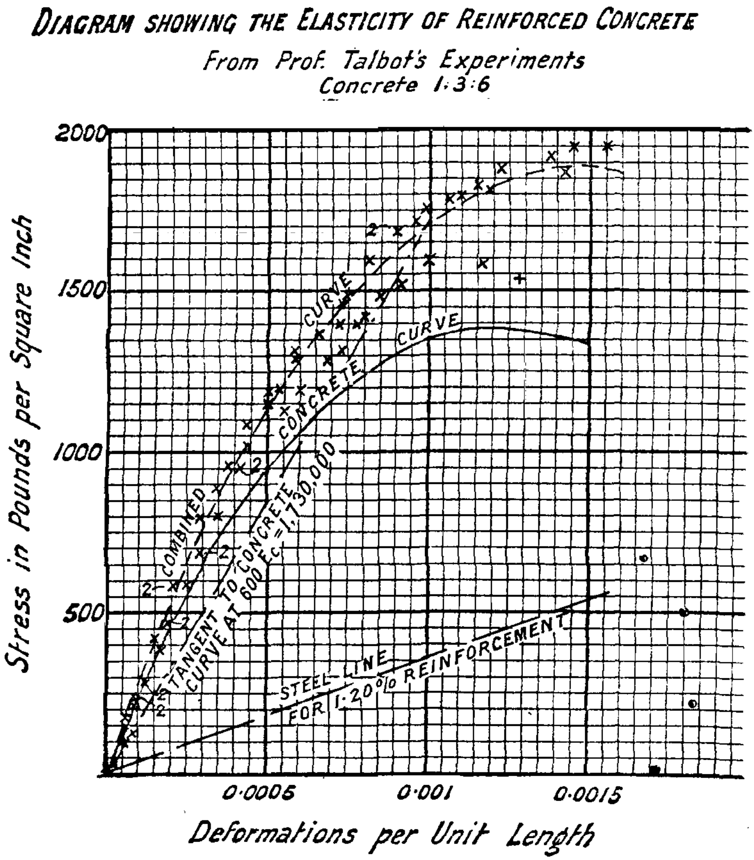


FIG. 4.

E_c the greater will be the added resistance of the reinforcement.

The load-deformation curve becomes flatter the more the member is loaded and unloaded, and consequently the value of E_c will decrease, but the modulus of elasticity of concrete increases with age, and although these tendencies may balance one another, it seems probable that on the whole the value of E_c increases.

When enquiring into the value to be assumed for E_c, the effect of the conditions obtaining during the process of hardening and that of the permanent set taken by the concrete under repetition loading, must also be taken into account.

The effect of the conditions of setting upon the reinforcements and the concrete surrounding them are demonstrated by some experiments carried out by Messrs. Emerson and Peabody at the Case School of Applied Mechanics, which show that when a reinforced concrete specimen sets in air the concrete contracts, inducing compressive stresses in the reinforcements; while the reaction due to the elastic resistance of the steel bars places the surrounding concrete in tension.

These experiments showed that, in the case of 1 : 2 : 4 broken stone concrete hardened in air, the compressive stress on the steel eighty-four days after moulding was more than 3,000 lbs. per square inch, while in the case of gravel concrete the average stress in two specimens was about 5,000 lbs. per square inch.

From some experiments of the same nature on mortar specimens in the proportion of about 1 : 2·7 reinforced with iron bars and hardened *in air*, M. Considère found that the mean compressive stress in the metal was about

2,850 lbs. per square inch; while the mean tensile stress in the concrete was about 155 lbs. per square inch, sixty-three days after moulding.

For pieces hardened under water the stresses are reversed, but are not so great.

The American experiments showed that for 1 : 2 : 4 broken stone concrete specimens, eighty-four days after moulding, the tensile stress on the metal was about 3,000 lbs. per square inch; while for the gravel concrete the stress was about 1,500 lbs. per square inch.

M. Considère's experiments on 1: 2·7 mortar, sixty-four days old, show a mean tensile stress of 1,710 lbs. per square inch on the steel and a mean compressive stress of about 100 lbs. per square inch on the concrete.

The percentage of metal was 0·4 per cent. for the American experiments and 5·5 per cent. in those of M. Considère, the sectional area of concrete for the American experiments being 64 square inches and for M. Considère's experiments 2·4 square inches.

With regard to the effect of the permanent sets taken by the concrete under loading: When a pressure is applied to a reinforced concrete member the concrete and steel are shortened the same amount, but the steel would resume its original length on the removal of the load, whereas the tendency of the concrete is to take a permanent set. The two materials, however, are acting together, and the bond of the concrete on the steel is absolute under working stresses, consequently the steel in trying to regain its original length drags back the concrete, placing it in tension, while the concrete, by holding back the steel, induces a compressive stress in the bars.

The effect of this is that, under a fresh application of the load, the tension on the concrete must be overcome before it offers any compressive resistance, while the steel is already under compression, and consequently must be compressed to a greater extent than formerly under the same load. It therefore appears, taking all things into consideration, that while the value of E_c probably increases with the age of the member, the effect of this is in a certain measure counterbalanced when the member is hardened in the air, and for loadings after the first, by the greater load which the concrete will bear before its resistance equals the working stress allowed, while the resistance of the steel is greater for the same deformation, as it is already under compression before the load is reapplied.

When the member is hardened under water the effect of the conditions while hardening and that of the previous applications of the load will in some measure react on one another, but the stresses induced when the hardening takes place under water are not as great as those when the piece hardens in air, and it is very seldom in practice that a member is kept sufficiently moist to approximate the condition of hardening under water.

There seems, therefore, every reason to consider that we may safely adopt the value of 2,000,000 for E_c at an age of two or three months, when designing a compression member, and similar reasoning applies in the case of a member subjected to bending. This value of 2,000,000 for E_c gives a ratio of $\frac{E_f}{E_c} = 15$.

The results obtained from the numerous experiments which have been carried out by various authorities justify the assumption of a value of 15 for the ratio $\frac{E_f}{E_c}$, although it is probable from experimental research that $17\frac{1}{2}$ is more nearly the proper value. The value of 15 is now very generally adopted both by designers of reinforced concrete structures and such authorities as Governments, Societies, and Committees who have issued rules, regulations or recommendations on the subject. Some few of these have assumed values as low as 10, while others have gone as high as 20, but by far the greater number have accepted 15 as the value of this most important coefficient.

Hooped Concrete.

At the present time the method of resisting the compressive stresses on a member by hooping, is being largely employed, and rightly so, as the hooping strengthens a member considerably.

The first failure of any material under pressure is not an absolute crushing, but a bursting out, due to the shearing induced by the combination of the longitudinal compression and lateral swelling of the piece. This is clearly shown by the photographs (Figs. 5, 6 and 7), showing the failure of cast iron and stone specimens.

M. Considère was the first to appreciate the advantage of supplying resistance to the swelling of a piece under compression, and the experiments he carried out are most interesting.

It is evident that if a compression member is surrounded with bands or spirals, these will be put into tension when the lateral swelling occurs and will resist such swelling.

With regard to this, however, by far the greater proportion of the shortening occurs under high pressures above those which would be adopted as working stresses. Also in the case of hooped concrete, it is economical to employ steel with a high elastic limit for the longitudinal reinforcements, as we can obtain the full working resistance from the steel in the longitudinal bars, when the member is hooped.

Both M. Considère and Professor Talbot are agreed on the large lateral deflection to which a hooped concrete member may be subjected without failure, and point out that this, together with the scaling off of the concrete outside the hoopings long before failure is approached, are great safeguards, as warning signs, occurring considerably before there is any danger of failure.

After the outer shell of concrete has peeled off the column has still a very considerable resistance. In one of the columns tested by M. Considère (the same as mentioned before) the load when the outer skin of concrete peeled off was 3,410 lbs. per square inch, whereas the ultimate loading was 12,690 lbs. per square inch, about $3\frac{3}{4}$ times greater.

Professor Talbot's results on full-sized columns, though very remarkable, were not so high as those obtained by M. Considère, in his experiments on small specimens.

The concrete used by Professor Talbot was mixed in the proportions of 1 : 2 : 4, or that usually employed in structures of reinforced concrete; the columns were 10 feet long and generally 12 inches in diameter. Some were bound with hoops of steel 1 inch wide and of various thicknesses, while others were spirally wound.

One column was twice loaded to 2,000 lbs. per square inch, having then a unit shortening of 0·0041. After this loading the spirals were removed and the plain concrete gave an ultimate resistance of 1,080 lbs. per square inch. Professor Talbot gives it as his opinion that the plain concrete before it had been tested would have had a resistance of 1,200 lbs., and consequently, after twice loading the spirally wound column up to 2,000 lbs. per square inch, the concrete, when the spirals had been removed, still retained 90 per cent. of its original strength.

A similar experiment was carried out by M. Considère on concrete mixed in the proportions of 840 lbs. of cement to a cubic yard of gravel, or about 1 : 3.

This cylinder when hooped bore a load of 6,970 lbs. per square inch, with a unit shortening 0·006, and after the removal of the spirals the plain concrete gave a compressive resistance of 1,420 lbs. per square inch.

Another cylinder mixed in the proportions of 630 lbs. of cement to 1 cubic yard of gravel, or about 1 : 4, withstood a pressure of 10,270 lbs. per square inch, with a unit shortening of 0·024 ; while after the spirals were removed the concrete failed at a pressure of 925 lbs. per square inch.

It appears from the various experiments that the hooping of a compression member gives greater security against sudden failures and unevenness in the concrete and will allow a higher working stress to be safely allowed for the concrete than is the case when longitudinal bars only are used tied together by bindings spaced some distance apart, as is usually the case.

It is also evident, that as in the case of members reinforced with longitudinal bars in the ordinary way, we may safely allow a working unit resistance for the longitudinal bars fifteen times greater than the working unit resistance allowed for the concrete, and if such allowance exceeds the safe resistance of mild steel the employment of high carbon steel will enable full use to be made of the longitudinal bars.

Conclusions.

The conclusions which may be drawn from the behaviour of concrete and reinforced concrete under compression may be briefly stated as follows:—

(1) That concrete mixed in the proportions of 630 lbs. of cement to 13½ cubic feet of sand, and 27 cubic feet of stone, or as expressed by volume 1 : 2 : 4, concrete may be safely assumed to offer a resistance of 500 lbs. per square inch when subjected to direct compression and reinforced by longitudinal bars and bindings some distance apart, and 600 lbs. per square inch when subjected to flexure.

For weaker or stronger concrete the strength may be considered to vary directly as the proportion of cement in the mortar, provided that the amount of stone in the concrete is not excessive.

The strength of concrete is not increased by reducing the proportion of stone below a certain amount depending somewhat upon its composition, or, in other words, the strongest concrete is that produced when the mortar exceeds the voids in the stone, after ramming, by about 5 per cent.

(2) That concrete when hooped will withstand considerably greater pressure than when reinforced with longitudinal bars with bindings some distance apart. The proper

safe resistance to allow for hooped concrete is still somewhat in dispute, but an increase of 50 per cent. on the resistance allowed for concrete reinforced in the ordinary manner is allowed by the most conservative authorities, and it is extremely probable that with efficient hooping $2\frac{1}{2}$ times the compressive resistance allowed for pieces reinforced in the ordinary way could be safely permitted.

(3) That whereas steel is truly elastic under high stresses, concrete has practically no truly elastic phase, and the nature of its composition renders it difficult to assess any very definite value for its instantaneous modulus of elasticity under any defined stress.

As a consequence, although steel may be assumed to have a modulus of elasticity of 30,000,000 lbs. per square inch, the value of the modulus of elasticity of concrete at a stress of from 500 to 600 lbs. per square inch can only be approximately assumed to be about 2,000,000 lbs. per square inch. Fortunately the effect of a considerable variation in this coefficient has no very marked effect on the results of the calculations.

The internal stresses set up in the steel and concrete due to the conditions of hardening in air and the permanent set taken by concrete under small loads tend to counterbalance the effect of the increase in its coefficient of elasticity due to the combined effects of repeated loadings and ageing.

(4) That it appears advisable to allow compression members to harden in air when reinforced with longitudinal bars and bindings at some distance apart; but that hooped compression members will offer greater resistance when kept very damp, or, better still, allowed to harden under water.

Behaviour under Flexure.

The behaviour of reinforced concrete when subjected to flexure is more complicated than when under simple compression, for such pieces there will be tensile strains on the concrete and steel, while the shearing stresses have greater influence. The adhesion between the concrete and the metal becomes a factor of considerable importance, while the elastic behaviour of the concrete in compression is again a factor affecting the whole fabric of the design.

Stress Area.

The behaviour of the concrete and reinforcement under compressive stress has already been discussed, but in the case of the design of members under flexure it becomes essential to study the nature of the stress-strain relation under gradually increasing stresses, or, in other words, to make an extended investigation into the form of the stress-strain curve, since, as the strain varies uniformly from the neutral axis to the outer fibre, the resistance must vary from fibre to fibre with the strain, and with it the total compressive resistance of the portion of the beam between the neutral axis and the outer fibre.

The relation of strain to stress in concrete is shown by diagrams (Figs. 3 and 4), which were referred to when dealing with its modulus of elasticity, and these diagrams show the effect which the non-elastic qualities of concrete has on the true stress area on the compressive side of a reinforced concrete beam.

To demonstrate this the load-deformation or stress-strain curve for the concrete may be plotted in diagram

(Fig. 8,) with the difference that in this case the deformations have been plotted as ordinates and the stresses as abscissæ. The curve in this figure is a fair average curve for a concrete in about the proportions of 1 : 2 : 4.

The diagram so produced represents the resistances in the fibres of the concrete just prior to failure, as the deformation gradually increases from *nil* at the neutral axis to a maximum at the outer fibre. The curve is parabolic in form and approaches so nearly to a true parabola that it can be calculated as a parabola without appreciable error.

Diagram Fig. 8 has been plotted from Professor Talbot's experiments, from which he has clearly shown that the stress-strain curve of concrete in compression is for all practical purposes a parabola.

From the results of fifteen tests of plain concrete columns made by Professor Talbot, the average maximum observed stress was 1,929 lbs. per square inch, whereas the average maximum stress obtained from a truly parabolic curve was 1,940 lbs. per square inch.

The parabolic nature of the stress-strain curve has also been demonstrated by Professor Hatt, of Perdue University, U.S.A., who studied a large number of experiments carried out at the Watertown Arsenal, and found that the average of the ratio of the area of the stress-strain diagram of the concrete to the area of the surrounding rectangle was 0·657, whereas the same ratio for a truly parabolic curve was 0·666, and that the ratio of the moment of the stress-strain diagram of the concrete to the moment of the area of the surrounding rectangle was 0·453, while the same ratio for a truly parabolic curve is 0·417.

The curve shown in Fig. 8 is that obtained when the outer fibre is offering its ultimate resistance, and cannot be taken as the stress-strain curve under any other conditions; but Professor Talbot has deduced formulæ for obtaining the stress area of the concrete under various phases of loading.

The stress area when the outer fibre suffers any less

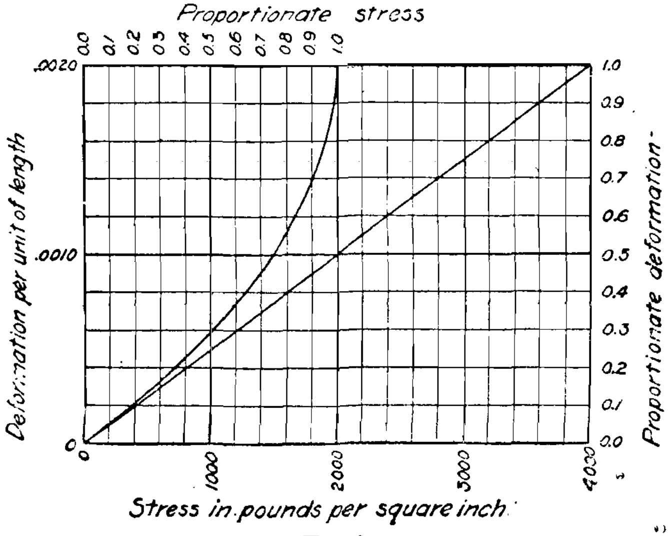


FIG. 8.

deformation than the ultimate is easily obtained graphically by cutting off the diagram at the required deformation, and the stress-strain curve being parabolic, we can easily calculate the stress in the outer fibre at various deformations below the ultimate.

The diagram (Fig. 8) also shows the ratio of the resistance developed in any fibre to the ultimate resistance at any ratio of deformation to the ultimate deformation and *vice versâ.*

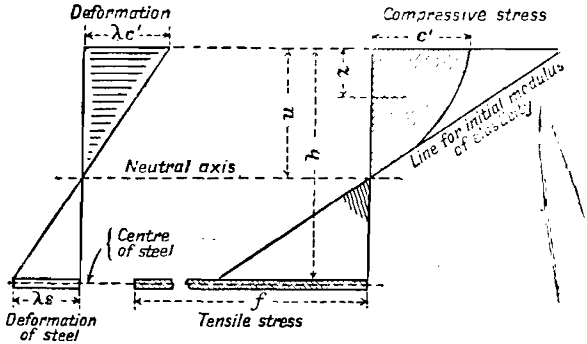


FIG. 9.—Stress and deformation at ultimate deformation.

Consequently, if in the diagram the origin of the curve is considered as the neutral axis of a beam and the outer

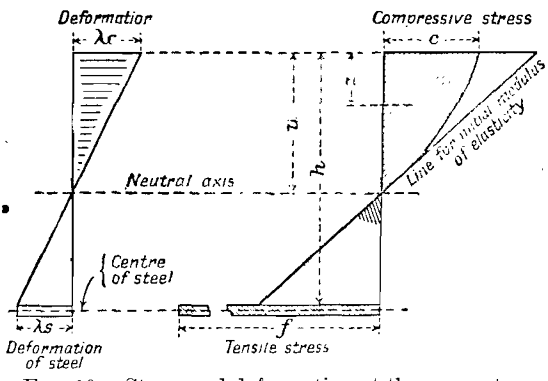


FIG. 10.—Stress and deformation at three-quarters ultimate deformation.

compressive fibre be taken at any parallel to the stress axis or the horizontal base, the ratio of deformation in this fibre to the ultimate deformation is given on the right side of the diagram, and the ratio of resistance of the fibre to the ultimate resistance is shown at the top of the diagram.

In other words, if we assume the outer fibre halfway between the horizontal base and the top of the diagram, or as having a deformation of one-half the ultimate, the resistance of this fibre is three-fourths the ultimate resistance of the concrete, and similarly for any other fibre.

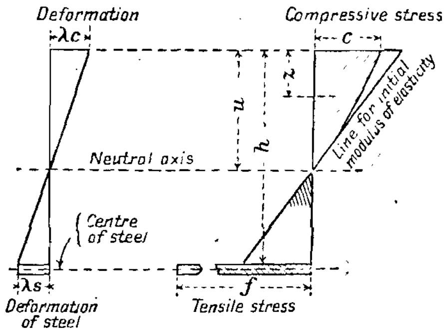


FIG. 11.—Stress and deformation at one-half ultimate deformation.

From a property of the parabola, if the modulus of elasticity were the same under all deformations, the ultimate stress would be twice that which is actually obtained when the stress-strain curve is a parabola. This is shown on the diagram by the straight line tangential to the parabolic curve at the origin.

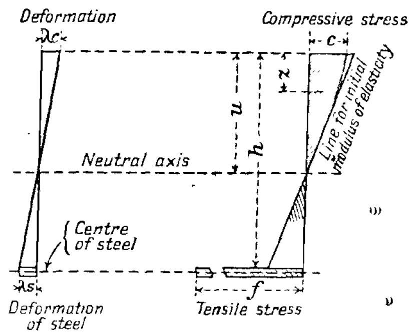


FIG. 12.—Stress and deformation at one-quarter ultimate deformation.

Diagrams (Figs. 9, 10, 11, and 12) show the stress areas in beams when the outer fibre in compression is subjected to the ultimate, three-fourths ultimate, one-half ultimate and one-fourth ultimate deformations.

Professor Talbot calculates the properties of the stress-strain diagrams from the initial modulus of elasticity and the deformation of the outer fibre in compression.

He obtains the values given in Table II., for a unit width, where λ_c is the deformation of the outer fibre, E_c the initial modulus of elasticity, u the distance of the neutral axis from the outer fibre in compression, and c the resistance of the outer compressive fibre.

The last column gives the values for a straight line stress-strain curve, for the purpose of comparison.

TABLE II.

Property.	At ultimate deformation.	At $\frac{3}{4}$ ultimate deformation.	At $\frac{1}{2}$ ultimate deformation.	At $\frac{1}{4}$ ultimate deformation.	Straight line stress-strain curve.
c	$\frac{1}{2} E_c \lambda_c$	$\frac{5}{8} E_c \lambda_c$	$\frac{3}{4} E_c \lambda_c$	$\frac{7}{8} E_c \lambda_c$	
Stress area	$\frac{2}{3} c u$	$\frac{3}{5} c u$	$\frac{5}{9} c u$	$\frac{11}{21} c u$	$\frac{1}{2} c u$
Distance of the centre of gravity of the stress area from the neutral axis.	$\frac{5}{8} u$	$\frac{23}{36} u$	$\frac{13}{20} u$	$\frac{29}{44} u$	$\frac{2}{3} u$

It will be seen from the above that at one-fourth the ultimate deformation the parabolic stress-strain curve approximates very closely to a straight line, in which case the stress area would be a triangle and equal $\frac{1}{2} c u$, while the distance of the centre of the compressive resistances from the neutral axis would be $\frac{2}{3} u$. This is also very evident from the diagrams (Figs. 8 and 12), where it is seen that the parabolic curve is very close to the tangent at the origin when the deformation is 0·25 of the ultimate. The resistance at one-fourth the ultimate deformation is about 0·45 the ultimate resistance, whereas the maximum safe allowable resistance used in conservative design is about one-fourth the ultimate.

It will be seen, therefore, that although Professor Talbot is perfectly correct in his theory based on a parabolic stress-strain curve, if the calculations are based on the ultimate resistance, still in practice, when calculating from the safe working resistances, the stress area in compression is very nearly triangular.

In America it is very usual to calculate using the resistance at the elastic limit of the steel and a resistance very near the ultimate of the concrete and to increase the bending moment by way of a factor of safety. Under such conditions it is perfectly legitimate and more accurate to consider the stress area as parabolic, but in this country it is usual to calculate using the working resistance, in which case a triangular stress area is to all intents and purposes correct.

If the resistance of the steel employed in the design is taken as its elastic limit, the resistance of the concrete should be taken as three-fourths the ultimate, or that obtained for a deformation of one-half the ultimate, as Professor Talbot has found from the results of many experiments that the stress deformation curve for concrete takes a decided bend at one-half the ultimate deformation, clearly indicating this as the phase on which the resistance should be based.

Bond.

The resistance of a reinforced concrete piece when subjected to flexure depends entirely on the bond between the steel and concrete. If the steel once commences to slide through the concrete the effect is exactly the same as if the bottom flange of a steel girder were suddenly severed entirely from the web.

Fortunately the bond between concrete and steel is ample to resist the tendency to sliding, even when the bars are plain round or square commercial shapes.

Many experiments have been carried out to ascertain the bond resistance, and the results obtained vary considerably This is not surprising, since many conditions exist which have a controlling influence—particularly the roughness of the bars, the density of the concrete, the amount of water used in mixing, the depth of embedment, and the age of the specimen.

Such experiments as these are necessarily conducted by the direct pulling or pushing out of the rods, but this hardly represents the true state of affairs in a beam where the concrete and steel are subjected to other than direct resistance to longitudinal parting.

It appears from tests that the bond resistance varies with the diameter or size of the bars, and that when the embedded length is sufficient no failure of bond occurs until the bars are stressed beyond their elastic limit, when they contract in area.

Table III. gives the bond resistance for plain mild steel bars found by some of those who have carried out tests :—

TABLE III.

Authority.	M. de Joly.	M. Considère.	Coignet and Tideseo.	Prof. Marburg.	Prof. Talbot.	
				Average.	Average. 1905	1906
Lbs. per square inch of surface.	286 to 686	256 to 500	285 to 335	253	281	424

TABLE III.—*continued.*

Authority.	Prof. Spofford.	Prof. Hatt.	Prof. Warren.	Prof. Van Ornum.
Lbs. per square inch of surface.	219 to 274	636 to 756	Average. 198	Average. 150

These results were obtained under very varying conditions in the density of concrete, length of embedment, diameter of bars, etc., and show the variation of results which have been obtained.

Slightly rusted bars naturally offer greater resistance to sliding than smooth bars.

It has always been evident that the bond resistance is in a great measure due to the grip of the concrete on the steel and not altogether to the true adhesion, as the outer layers of the concrete will harden more rapidly than the inner layers, and since concrete setting under ordinary conditions will contract, this causes a tightening of the concrete around the steel.

Recent experiments have demonstrated that this is the case, and that after the bars had commenced to slide in the concrete there is still a considerable frictional resistance amounting to large percentages of the original bond.

Professor Hatt, using plain round rods and 1 : 2 : 4 concrete, found the average initial bond resistance for $\frac{7}{16}$ inch rods to be 636 lbs., and for $\frac{5}{8}$ inch rods 756 pounds per square inch of surface, while the frictional resistance after the rods had begun to move was from 60 to 70 per cent. of the initial bond.

Professor Talbot, from numerous experiments, found the frictional resistance, after the bars had moved about $\frac{1}{4}$ inch, amounted to from 54 to 72 per cent. of the initial bond in the case of mild steel plain round bars, and 32 to 49 per cent. of the initial bond in the case of cold rolled shafting.

Professor Van Ornum tested eighteen specimens, using plain square bars, and found that the average frictional resistance after the bars had commenced to slide was 67 per cent. of the initial bond.

The conditions under which failure by sliding of the reinforcements is most likely to occur would evidently be where the structure is subjected to severe and frequent shocks, and in such cases it would appear advisable to use some form of bar shaped to give a mechanical bond, although from the results of some tests made by Professor Van Ornum on thirty specimens subjected to severe treatment of this nature the bond resistance was only reduced 20 per cent. The specimens tested by Professor Van Ornum were subjected to vibrations by means of a machine devised so as to give a series of blows on the concrete in which the bars were embedded, while the specimen was supported by the bars. The blows were delivered at the rate of 150 per minute, and each blow imparted about 740 inch-lbs. of work to the specimen. After being subjected to an average of 50,000 blows each, the rods were pulled out in the ordinary manner.

As might be expected, flat bars give a much lower resistance to sliding than either round or square, and square bars as a rule give a lower resistance than round.

Bars, specially formed so as to offer increased resistance against sliding, have been introduced in America and this country, and at first sight would appear an immense improvement on ordinary round or square bars. Experiment and practice have, however, demonstrated conclusively, that the bond between plain bars and concrete offers more resistance against sliding than will be required under ordinary circumstances, if the bars are extended well beyond the supports of a beam and are bent out at right angles for a length of about 2 inches, or, better still, bent to a hook having a radius of 1 to $1\frac{1}{2}$ inches. The bars are sometimes split and opened at the ends, but it is better to bend them over.

The longitudinal shearing stresses are greatest at the end of a beam and decrease to *nil* at the centre of the span, consequently if plain round or square bars are firmly secured at the ends they cannot slide through the concrete unless the cracks on the tensile side extend beyond them.

In tests of beams reinforced with mild steel round or square bars, the first failure is never due to the slipping of the rods. Professor Talbot, in his 1905 tests, calculated the bond stress to be as high as 193 lbs. per square inch in one case, and yet there was no evidence of slip.

It appears that the safe bond resistance for plain round or square bars may be taken as 100 lbs. per square inch of surface. If it is found that a greater resistance is required, then provided the reinforcements in vertical planes are rigidly attached to the longitudinal bars so as to transmit the diagonal tensile stress direct to these; no slipping can occur.

If the reinforcements in the vertical planes are not rigidly attached to the longitudinals, and the bond stress is found to exceed 100 lbs. per square inch of surface, then special bars having a mechanical bond may be employed.

Shearing and Diagonal Tension.

The shearing resistance of concrete until a short time ago was always supposed to be small, but recent investigations, especially those of Professor Talbot, have shown that its value is considerably higher than was supposed.

It is very difficult to test for direct shear, as there is always a probability of the results being influenced by bending, but there is no doubt that the shearing resistance of concrete is at least 50 per cent. of its compressive resistance.

Professor Talbot's tests on plain flat pieces supported on a plate having a circular opening 6 inches in diameter, the die producing shear being $5\frac{7}{8}$ inches diameter, gave an average shearing resistance of 67 per cent. of the compressive resistance of cylinders made of the same concrete, the *maximum and minimum values being 49 and 83 per cent.*

The failure of these plates showed signs of tensile stresses having been induced.

Plain blocks recessed gave better results, the average shearing resistance being 72 per cent. of the compressive resistance with maximum and minimum values of 86 and 52 per cent.

Other recessed blocks reinforced around the shearing area, to further resist the bursting pressure, gave an average shearing resistance of 122 per cent. of the compressive resistance with minimum and maximum values of 139 and 88 per cent.

Restrained beam tests, in which the beams were firmly held at the ends, leaving a length of $4\frac{1}{8}$ inches on which the pressure was applied over a length of 4 inches, gave an average shearing resistance of 82 per cent. of the compressive resistance with maximum and minimum values of 100 and 58 per cent.

The shearing resistance, when compared with the compressive resistance of cubes instead of cylinders, gave lower results.

These tests show that the direct shearing resistance of concrete is considerable and may in some cases be greater than its compressive resistance, but when a reinforced concrete beam is subjected to flexure, although the resistance of the concrete to tension is neglected in the calculations, it is nevertheless subjected to considerable tensile stresses.

The tensile stresses above the rods are combined with those due to direct shearing, and this combination forms the diagonal tension which must be resisted either by the concrete itself or by special reinforcements.

It is difficult to compute the exact amount of the diagonal tensile stress, since at any section it is dependent on the horizontal tensile stress developed in the concrete at that section. We can, however, calculate it with reasonable accuracy by considering only the direct shearing stresses and allowing a resistance for the concrete considerably below its safe resistance to direct shearing.

The direct shearing resistance of concrete is probably in the neighbourhood of 75 per cent. of its compressive resistance. Concrete in the proportion of 1 : 2 : 4, or 630 lbs. of cement, to 13½ cubic feet of sand and 27 cubic feet of stone, would therefore have an ultimate shearing resistance of about 1,800 lbs. per square inch. For the purposes of calculation, however, we must assume a value considerably less than this for the reason already stated.

Professor Talbot found, from experiments carried out during 1905 on beams reinforced with longitudinal rods only, that, in those which failed by diagonal tension the apparent ultimate shearing resistance, as calculated, was on an average 123 lbs. per square inch.

For *the safe resistance to be used in the calculations we may assume a value of* 60 *lbs. per square inch, such calculations being based on the direct shear.*

Tensile Strain on the Concrete.

Although the tensile resistance of the concrete is generally neglected in the calculations for reinforced concrete beams, tensile stress must be developed, and as the cracking of the concrete to any noticeable extent will allow moisture to reach the reinforcing bars, it is well to be assured that in properly designed beams no such fissures will occur.

M. Considère, in his well-known experiments on specimens carefully prepared and stored in water, found that the concrete on the tensile side of one of the test pieces did not crack when the elongation was $\frac{1 \cdot 98}{1{,}000}$ of the length. In another case, after subjecting a specimen to flexure, and producing an elongation of $\frac{1 \cdot 27}{1{,}000}$ in the extreme fibre under tension, two small cracks, $\frac{1}{12}$ and $\frac{1}{6}$ inch long, were observed.

After being subjected to these elongations, the concrete on the tensile side of the reinforcing bars was sawn away, and this portion bore handling without breaking, and moreover was again subjected to flexure and offered considerable resistance before failure.

M. Considère concluded from the results of these tests that reinforced concrete would bear very much greater elongation without cracking than had been previously thought, and under these elongations would still retain its maximum resistance.

As in the case of metals, concrete will elongate to a considerably greater extent before failure when subjected to flexure than when tested in direct tension. This is probably due to the assistance offered by contiguous less stressed fibres.

The increased deformation of plain concrete under flexure, however, only amounts to about $2\frac{1}{2}$ times its deformation under direct tension, whereas M. Considère's experiments show that when reinforced concrete is subjected to flexure the deformation before failure will be about twenty times that of plain concrete under direct tension.

Some experiments carried out by M. Mesnager in 1903 on reinforced concrete specimens under direct tension showed an elongation of $\frac{1 \cdot 35}{1,000}$ before failure, which bears out M. Considère's conclusions.

M. Christophe, Professor Talbot, and Professor Turneaure carried out beam experiments at a later date to those of M. Considère's, from which they deduced conclusions antagonistic to his.

M. Christophe concluded that although, in a reinforced beam, the concrete in tension undoubtedly elongated before cracking to a greater extent than plain concrete would do under direct tension, still the elongation was considerably lower than that found by M. Considère. He also calculated the resistance offered by the concrete when the first crack showed, and found this to be about 45 lbs. per square inch, while it should have been about 170 lbs. per square inch according to M. Considère.

Professor Turneaure carried out some experiments on reinforced concrete beams during 1902–3. These beams were of 1 : 2 : 4 concrete, and were tested at an age of one month. Careful observations were made for the appearance of the first crack, the beams being moistened to show up fine hair cracks.

Professor Turneaure states that the elongations when the first fine hair cracks appeared were about $\frac{0{\cdot}1}{1{,}000}$ to $\frac{0{\cdot}2}{1{,}000}$, which agrees well with the elongations obtained from plain concrete, while cracks visible to the eye occurred with elongations of about $\frac{0{\cdot}35}{1{,}000}$, and concluded that M. Considère's conclusions were incorrect.

M. Considère, however, was careful to point out that small invisible cracks will frequently occur during the hardening of beams in air and before any load is applied, and he mentions the fact of the two cracks, $\frac{1}{12}$ and $\frac{1}{6}$ inch long, observed in his test. The fine cracks detected by Professor Turneaure were merely surface cracks, generally only about $\frac{1}{8}$ inch long, and did not extend across the whole width of the beam. According to the Professor's own statement, such cracks could have very little effect on the tensile resistance or the elongation before any crack sufficient to endanger the reinforcement is produced.

Professor Talbot, in his reinforced beam experiments carried out early in 1904, observed that no cracking occurred until after the concrete had elongated $\frac{0{\cdot}1}{1{,}000}$ of its length. After this elongation, and during what he describes as the readjustment stage, minute cracks probably existed, but were not easily detected. The elongation of the steel during this phase reached about $\frac{0 \cdot 15}{1,000}$, that of the outer tensile fibre of the concrete being considerably more, probably about $\frac{0 \cdot 2}{1,000}$.

Between this deformation and a deformation of the steel of about $\frac{1}{1,000}$, that of the outer tensile fibre of the concrete being probably over $\frac{1 \cdot 5}{1,000}$, fine vertical cracks appeared. Professor Talbot does not state at whatactual deformation these cracks first became visible, but it is probable that they were not of importance until the elongation of the outer fibre exceeded $\frac{1}{1,000}$.

In order to verify his former conclusions, M. Considère carried out further experiments in 1904 on two reinforced specimens, one of which was kept after moulding covered with bags and planks, which were frequently wetted, and the other was placed in water.

These were tested by flexure, the deformations being carefully measured.

The loading was stopped when the elongation of the extreme tensile fibres amounted to $\frac{0 \cdot 63}{1,000}$ for the beam stored under damp bags and planks and $\frac{1 \cdot 27}{1,000}$ for that stored in water.

On examination through a microscope after these elongations, no fissures could be detected, although a thin coating of cement had been applied to the tensile surfaces of the beams.

The lower portion of the beam containing the rods was carefully removed and slices of the beams, about $1\frac{3}{4}$ inches thick, immediately above the rods were then sawn off the remaining portion.

From calculation it was found that these slices had been subjected to elongations of from $\frac{0{\cdot}22}{1{,}000}$ to $\frac{0{\cdot}50}{1{,}000}$ in the case of the beam stored under damp bags, and from $\frac{0{\cdot}56}{1{,}000}$ to $\frac{1{\cdot}07}{1{,}000}$ in the case of the beam stored in water.

These slices were then tested under flexure, together with the remaining portion of the beam which had been above them.

It was found that the slices just above the rods offered very nearly as great resistance as the upper portions of the beams.

It therefore appears that M. Considère's former conclusions were amply verified.

M. Considère accounts for the hair cracks frequently noticed in reinforced concrete beams in the following manner. When concrete is exposed to dry air after its moulding, it is subjected to considerable contraction during the first few days, while its powers of resistance are small.

This contraction would be resisted by the reinforcements, but the concrete has not attained sufficient strength to induce tension in the reinforcement, nor has it sufficient ductility to adapt itself to the induced stress from the reinforcements.

Cracks are consequently produced, invisible at first, but which lengthen and open when a load is applied.

If, on the other hand, the concrete is kept moist for a sufficient period it does not contract, and consequently there is no tendency for the production of cracks until it has attained sufficient strength and ductility. The concrete will tend to contract directly it is allowed to dry, but by that time it has attained sufficient strength to withstand considerable elongation, and it will not crack in spite of the reaction of the reinforcements.

It appears evident, therefore, that all pieces which will be subjected to flexure should, if possible, be kept damp during the first few days after moulding.

It also appears safe to allow that the concrete will suffer on elongation of $\frac{1}{1,000}$ of its length before any crack can be sufficiently enlarged to endanger the reinforcement.

The elongation allowed by M. Considère in his calculations is $\frac{1 \cdot 5}{1,000}$, but this is perhaps unnecessarily high.

It can be shown that the concrete in a reinforced beam, using mild steel with a safe resistance of 16,000 lbs. per square inch, cannot be subjected to an elongation of $\frac{1}{1,000}$ unless it is grossly overloaded or has too much cover of concrete below the reinforcements.

The reason for this is that, unless high carbon steel is used for the reinforcements, the elongation which would produce the safe working stress in the steel cannot cause a deformation of $\frac{1}{1,000}$ at the outer surface of the concrete unless the depth of cover of concrete below the bars is considerably greater than is reasonably necessary or the stress on the concrete in compression is altogether excessive.

This is one reason against the use of high carbon steels. It is evident, for instance, that if a high carbon steel is employed, with a safe resistance of 30,000 lbs. per square inch and a modulus of elasticity of 30,000,000 lbs. per square inch, the elongation to produce a stress of 30,000 lbs. per square inch will be $\frac{30,000}{30,000,000}$ or $\frac{1}{1,000}$; and as the elongation of the outer tensile fibre of the concrete must considerably exceed that of the steel, it is clear that the use of such steel would be suicidal, and even with a safe limit of 20,000 lbs. per square inch dangerous elongations may be induced in the concrete and especially in pieces of small depth.

The existence of initial cracks appears to have no appreciable effect on the resistance of a beam. Professor Talbot made some special experiments on beams with artificial cracks and open spaces in the concrete on the tensile side. On comparison *of the deformations* of these beams with those of normal beams, there was no noticeable difference, and even the open spaces did not appear to be a source of weakness.

T Beams.

Special reference must now be made to the behaviour of T beams and the conclusions which may be deduced therefrom.

If a T beam is properly designed and the rib has a sufficient width, the failure will be of the same nature as that of a rectangular beam. There is, however, a marked tendency for the concrete to shear near the ends on the horizontal plane through the rib along the under side of the slab, especially if the rib is of insufficient width, and it is consequently advisable to introduce shearing reinforcements extending from the slab into the rib for some distance at the ends of a T beam and on each side of any supports.

A failure by diagonal shearing is more likely to occur in a T beam than in a rectangular beam, as a T beam is calculated as having an increased area resisting the compressive stresses due to the extra width allowed for the slab, whereas the diagonal tensile stresses are only resisted by the width of concrete on the rib. In T beams, therefore, it is nearly always necessary to place special reinforcements to resist diagonal tension.

Conclusions.

To recapitulate the main conclusions to be drawn from the behaviour of reinforced concrete under flexure:—

(1) The first signs of failure in a properly designed piece is on the tension side of the neutral axis, and is due to direct or diagonal tension.

(2) The initial cracks in the concrete have no effect on the resistance of the piece.

(3) The cracks first formed on the tension side gradually extend into the compression side under relatively heavy loading, which indicates a rise in the position of the neutral axis as the load increases.

(4) As the failure proceeds there is a slight slipping of the longitudinal reinforcements through the concrete, due

to the contraction of the area of the metal in tension, unless it is of such a character as to give a mechanical bond.

(5) The longitudinal reinforcement in tension bends when the cracks producing failure open sufficiently.

(6) The surface of concrete under compressive stresses is the last portion of the beam to show signs of failure; this flakes off in horizontal layers.

(7) In some cases the first sign of failure is that by diagonal tension (frequently referred to as shearing) near the supports. This is indicated by inclined cracks near the supports sloping outwards and upwards and starting at the commencement of the failure from the top of the bottom longitudinal bars—A failure beginning in this manner often terminates by a splitting along the top of the longitudinal bottom bars towards the support, caused by the load on the outer portion of the beam, beyond the inclined crack, being transmitted to the lower part of the beam (containing the longitudinal bars) near the support.

(8) The strength of reinforced concrete under flexure increases with age, but in a lesser degree than its resistance to direct compression, clearly indicating that the resistance to compression is not the ruling factor in the failure.

(9) The concrete in compression may be allowed a maximum safe resistance of 600 lbs. per square inch if mixed in the usual proportions of 1 : 2 : 4, or about 630 lbs. of cement, to 13½ cubic feet of sand and 27 cubic feet of stone; while weaker or stronger concretes will have a proportionately reduced or increased resistance varying in direct proportion to the proportion of cement in the mortar used to make the concrete (*vide* Table I.).

(10) The safe resistance of mild steel in tension may be

taken as half its elastic limit, or for ordinary commercial steel 16,000 lbs. per square inch.

(11) Although the stress-strain curve for concrete is undoubtedly parabolic, this fact need only be taken into account, when determining the stress area of the concrete in compression, if a resistance is allowed approaching the ultimate, and the factor of safety is applied by assuming a greater bending moment than that which has to be provided for. For calculations based on safe resistances it is quite sufficiently accurate to assume a straight line stress-strain curve, making the stress area in compression triangular.

This of course necessitates the assumption, that a constant modulus of elasticity exists under stresses up to the maximum safe limit allowed, which, although not strictly true, is an assumption perfectly permissible from a practical standpoint.

(12) The failure is seldom if ever primarily due to the sliding of the reinforcements, and that although the resistance to sliding must be less in a beam, where the concrete is also resisting direct tensile stress, than in test pieces from which the bars are drawn out (the values found for this resistance are very variable), still the failure of a beam is seldom if ever primarily due to the sliding of the bars through the concrete. After the reinforcements have commenced to move there is still considerable frictional resistance between the concrete and steel.

Vibrations will not have an appreciable effect on the bond if the loads are such that the stresses in the materials are kept within safe limits. Although bars having a mechanical bond do undoubtedly offer greater resistance to sliding than plain round or square bars, still there appears no adequate reason for their employment on this account, unless in very special circumstances. Flat bars slide through concrete more easily than round or square bars. If plain round or square bars are extended well beyond the supports of a beam and are bent over or otherwise treated to resist sliding, a safe bond resistance of 100 lbs. per square inch of surface may be allowed, and this may be exceeded if the reinforcements in the vertical planes are rigidly attached to the longitudinals, although, if a rigid attachment does not exist, bars with a mechanical bond should be used if the bond stress is found to exceed the limit of 100 lbs. per square inch.

(13) The shearing resistance of concrete is very considerable, probably from 50 to 75 per cent. of its compressive strength, but what, until recently, was known as the shearing of concrete beams is in reality a failure by diagonal tension and is influenced in a considerable degree by the longitudinal tensile stress in the concrete, which is difficult to determine. It is therefore advisable to make the calculations for the direct shearing stress only, allowing a resistance of 60 lbs. per square inch for the concrete. In the case of T beams it is nearly always necessary to provide reinforcements to resist diagonal tension.

(14) The tensile resistance of the concrete should be neglected when designing a member subjected to flexure. Dangerous cracks on the tensile surface will not occur if mild steel reinforcement is used, but it is very necessary to ascertain the tensile deformation of the concrete if high carbon steel is employed, and this deformation must not exceed $\frac{1}{1,000}$.

(15) In the case of T beams it is advisable to insert reinforcements in the vertical planes near the supports, extending from the slab well into the ribs.

Arches.

The behaviour of arches shows that the stability of the abutments has a considerable effect on their resistance.

The failure of arches with a curved extrados is generally due to the opening out of the span caused by the abutments yielding, and that of an arch with a flat extrados by the top of the abutment breaking away and allowing the arch to drop at the centre.

As in the case of beams, the first failure in an arch occurs where the tensile stresses are greatest and, if the arch is properly designed, is never due to a lack of resistance to compression.

Since the stresses in an arch are to a considerable extent compressive, it is advisable to tie the reinforcements at the intrados and extrados well together.

In an arch with a flat extrados there is also a tendency for the reinforcement at the intrados to straighten out, consequently it should be well tied to the concrete or to the reinforcement near the extrados through the central portion of the span. The bars extending through the arch near the extrados should be continued well down into the abutments to prevent the upper part of the abutments breaking away.

In arched bridges with open spandrels formed with columns or arches there is a tendency to local shearing failures where the columns or piers of the spandrels rest on the arch, and consequently adequate reinforcements to resist shearing should be inserted in the arch at these places.

For large span arches it is advisable to place reinforcements along the whole length of the arch, both near the extrados and intrados. In small span arches with a curved extrados, under a uniformly distributed load, reinforcements should be placed near the intrados throughout the whole length, and also at the extrados extending from the abutments for a length of about one-fourth the span.

Arches with a flat extrados, should be provided with reinforcements placed near the extrados for a short distance from the abutments and then inclined downwards, passing through the central portion of the span near to the intrados. They should also have reinforcements both near the extrados and intrados, continuing through the entire span. The reason of this method of reinforcement in the case of arches with a flat extrados is that in reality they may be considered as acting in a similar manner to cantilevers, with a central span supported at their ends. This is also the reason for the necessity of bending down the reinforcements near the extrados, so as to extend well into the abutments.

Effects of Atmospheric Conditions.

There is one more question to be considered which applies generally to reinforced concrete structures, and that is the effect of changes in temperature or hygrometrical conditions.

Concrete expands with an increase of temperature or dampness and contracts with a decrease of temperature or dryness.

The changes produced by these conditions are considerable and are the cause of the unsightly cracks which so frequently appear in concrete structures.

If a concrete structure is held rigidly at its ends or if it is sufficiently long that it cannot expand or contract as one piece, it is obvious that severe stresses must be set up by atmospheric changes.

The rise in temperature or increase of dampness will not as a rule have a noticeable effect, as the concrete can withstand very considerable compressive stresses, but under a fall of temperature or excessive dryness tensile stresses are set up, and the concrete, being unable to resist these, will crack.

This tendency to cracking may be resisted by placing reinforcements near the exposed surface, and the amount of such reinforcements can be easily calculated.

It must be borne in mind, however, that the changes of temperature will act directly on the reinforcement as well as on the concrete, and consequently sufficient steel must be used to resist the stresses directly induced by the variation of temperature and also those induced by the concrete, and for this reason it is advisable to use a high carbon steel for this purpose. A mechanical bond is also a distinct advantage.

The concrete unaided cannot resist the tensile stresses induced by a contraction, when held firmly at the ends, whereas the steel is perfectly able to do so, and has sufficient resistance, if of the requisite sectional area, to give all the assistance necessary to the concrete.

CHAPTER III

NECESSARY ASSUMPTIONS FOR PURPOSES OF CALCULATION

It is necessary to assume certain hypotheses on which to base any calculations for obtaining the dimensions of reinforced concrete structures, and unfortunately it is by no means certain that these are absolutely true. The development of the most elaborate theories must depend upon these hypotheses, and as a consequence the scientific accuracy of the theories is to a great extent nullified at the outset.

It is better, however, to employ formulæ which have a scientific basis, although they may be derived by the use of hypotheses, some of which are not absolutely correct, than to use empirical calculations which frequently have no basis at all. There are, of course, practical formulæ founded on scientific theories, which it is safe to employ within certain limits, but unless the subject is thoroughly understood and the reasoning on which such formulæ are based and their limitations are entirely appreciated, it is better to avoid their use, since, while calculations of this nature are perfectly safe in the hands of those who are aware of the extent to which they may be adopted, they may become a source of great danger in the hands of the uninitiated.

It must be borne in mind, also, that all theories, even the most elaborate, are only approximate, being based on the best information that can be obtained. We are certainly justified in assuming hypotheses which may not be absolutely correct on which to base our calculations if we thoroughly recognise that the results obtained are not absolutely exact, and at the same time can satisfy ourselves that they are sufficiently correct for all practical purposes.

We will, therefore, take the hypotheses which it is necessary to adopt for the purposes of deriving formulæ for the design of reinforced concrete structures and study the effect of their inexactitude upon the calculations :—

That the applied forces are perpendicular to the neutral surfaces of pieces subjected to bending.

This is a general assumption which is accepted for all beams and girders, and although it is clearly incorrect, in consequence of the deflection of the piece, it is certainly sufficiently near the truth for practical purposes for cases of simple bending.

It cannot, of course, be applied to arches which are subjected to compression and flexure combined.

That each fibre of the concrete acts by itself, not being affected by the contiguous fibres.

This supposes that each fibre will be elongated or contracted by the stress to which it is subjected, as if it were alone.

It is almost certain that the fibres of a column or beam do not act in this manner and that there is what may be termed a "striction" between adjacent fibres which modifies the deformations controlling the action of the several fibres, so that they undergo less deformation under the induced stresses than they would under the same direct stresses in a testing machine.

This effect is the probable cause of the divergence of the theoretically calculated and actual strength of rectangular homogeneous beams, the breaking load on a cast iron beam being from two to three times the calculated load. Such a "striction" must exist, as it is entirely due to this that a beam is enabled to withstand the longitudinal shearing stresses which tend to cause the fibres to slide over one another. In the same way a column which is loaded locally certainly acts as a whole in resisting the stresses as in the case of a pin-jointed compression member.

The admission of the hypothesis is entirely on the side of safety, and may therefore be allowed, as its admission considerably increases the factor of safety and thus probably more than counterbalances the effects of the inexactitudes in other assumptions which may be on the side of danger.

That there is always a solid contact between the reinforcement and the surrounding concrete.

It is generally assumed that the concrete and the metal act together, or, in other words, that the concrete follows the deformations of the reinforcements.

This assumption is very necessary for the simplification of the calculations, since without it we should have no definite law connecting the relative deformations of the two materials, and the stresses would not be related to one another in the ratio of the moduli of elasticity. It is very doubtful if this hypothesis is a true one: at any rate when approaching failure, there is every reason to suppose that the concrete hangs back and does not follow the deformation of the reinforcement.

As failure is approached the reinforcements slide in the concrete, and consequently the two materials cannot act together.

Under working stresses, however, there is probably little or no hanging back of the concrete, and at any rate, the difference in the deformations is very slight. Consequently, for all practical purposes, this hypothesis may be accepted. This does not apply to calculations based on the ultimate resistances of the materials such as are sometimes used, but, even in this case, the factors of safety employed are sufficient to cover the errors involved by its adoption.

It must be admitted that there is an extreme possibility that the reinforcement and the concrete in contact with it are at no time equally stretched or compressed, and that the theory of elasticity cannot be applied with the same absolute truth to a heterogeneous material as it can to the structural metals. The concrete in itself is a heterogeneous material, and is subject to small voids and cracks even when carefully made; it has also a very different molecular construction to that of the metal in the reinforcements, and the sudden change from the comparatively large grained concrete to the close grained metal must have some effect on the deformations.

The great shearing stresses where the concrete comes in contact with the metal must also have a tendency to cause a displacement of the reinforcement in the concrete. There appears, then, every likelihood that there is a difference of deformation in the reinforcement and the surrounding concrete, and to consider no such difference as existing is an incorrect assumption; but, fortunately, such inequality of deformation can have only a very small effect on the accuracy of the calculations, especially those based upon the working resistances of the materials.

With respect to the influence of the shearing stresses where the concrete comes in contact with the metal, it may be well to point out that, the greater the ratio of the perimeter of the reinforcements to their sectional area the smaller will be these stresses, and consequently it is advisable to use a series of bars of small dimensions placed near together, rather than bars of large area placed far apart. Of course, when small bars are used placed near together, they must have sufficient concrete between them to convey the stresses.

It is very evident that when large sections, such as angles, tees, joists, or large sizes of bars are employed spaced far apart, the inequality of deformation will be greatly intensified.

Where the shearing stresses are small there is not so much reason for avoiding the use of large sections placed far apart, and consequently this method of reinforcement is quite rational for arches and compression members, although the calculations are made on the assumption of equal deformations of the reinforcement and the surrounding concrete.

For pieces subjected to flexure, however, it is well to bear in mind as an object to be aimed at, that reinforced concrete should be made as homogeneous as possible, and consequently the employment of small sizes of bars is an advantage from a theoretical point of view, whatever it may be from a practical standpoint; and that although large bars placed far apart are more economical than small ones placed near together, the truth of the calculations may be dangerously departed from if economy of material is given too great a weight when deciding on the sizes of bars to be adopted. For slabs, a good practical rule to adopt would be to make the maximum distance allowed from centre to centre of the bars no greater than six inches, or for slabs thicker than six inches no greater than the thickness of the slab.

That plane sections remain true planes during loading.

This hypothesis, generally spoken of as the conservation of plane sections, is perhaps the most important of those on which the calculations for pieces under flexure are based, since it furnishes what is practically the starting point from which the calculations are derived. It follows from the hypothesis of the conservation of plane sections that the deformation of any fibre is directly proportional to its distance from the neutral axis, and consequently that the stresses in any two fibres are proportional to their distances from the neutral axis, multiplied by the respective moduli of elasticity of the materials of which they are formed.

This hypothesis rests mainly on the two previously considered, and its truth stands or falls with them. We have seen wherein these former hypotheses are incorrect and it is very evident that the effect on plane sections of any inequalities in the relative movement of the fibres of the concrete itself or of the concrete and the metal must be very considerable.

In the case of pieces acting under direct compression it is seldom that the load is applied uniformly over the section, and this must have a considerable effect on the deformations. The load on such pieces must be always more or less local in application, and the effec of this on the conservation of plane sections must be considerable, at any rate in the neighbourhood of the areas of concentration.

In the case of pieces subjected to flexure the adoption of this hypothesis becomes almost imperative, as without it the calculations would be made extremely difficult, if not impossible. It has been always allowed in the case of metallic structures, although it is probably not an exact truth even in this case, as the existence of shearing strains must tend to cause the originally plane sections to take a curved form, since the paths of the combined direct and shearing stresses follow curved lines.

When reinforced concrete is under consideration, it is certain that the effect of the different elastic properties of the concrete and the metal, upon the behaviour of plane sections, must be considerable.

Professor Warren, of Sydney University, carried out some experiments for the purpose of determining the truth of this hypothesis. He tested ten beams, taking careful measurements of the deformations at two points on each side in the depth of the beam as well as at the extreme fibres.

The diagram (Fig. 13) has been plotted from the results of these experiments, and shows that the original plane sections do not remain plane after bending and that the deviations are greater as the bending moment increases.

It is certain, therefore, that the results of experiments cannot be shown as a proof of the truth of this assumption, but it appears, nevertheless, that if small sections of metal are used for the reinforcements and the calculations are made for the working resistances of the materials, it is sufficiently near the truth for all practical purposes and may be used as a basis for the calculations, although we must admit, that its use must to a limited extent affect their scientific exactitude.

That there will be no initial stresses in the structures before loading.

Initial stresses may be set up by the expansion or

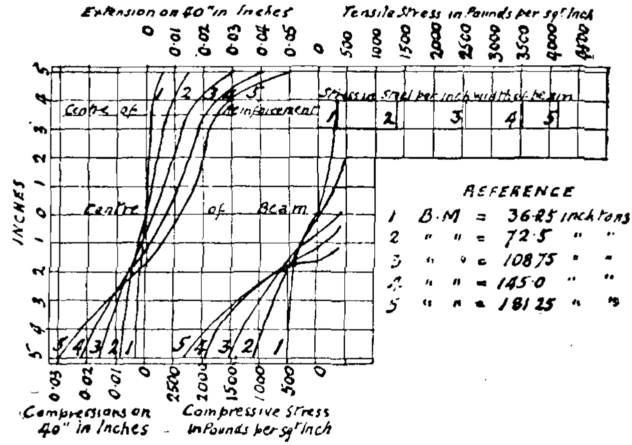


FIG. 13.

contraction of the concrete during setting, changes of temperature or the permanent deformations due to the load of the structure itself or previous superimposed loading.

It seems almost certain that initial stresses are set up by all of these, and it therefore becomes necessary to inquire into their nature and the effect they will have on the accuracy of the calculations based on the assumption that no such stresses exist.

Concrete hardened in air contracts, while if it is allowed to harden in water it will expand, consequently for pieces in which the concrete hardens under water tensile stresses are developed in the reinforcements and compressive stresses in the concrete; while when the hardening takes place in the air compressive stresses are set up in the reinforcement and tensile stresses in the concrete.

From M. Considère's experiments (page 34) it appears that the initial tensile stresses in the concrete due to its setting in air will very nearly reach the ultimate resistance in tension of similar pieces of plain concrete at the same age.

As the resistance of the concrete is neglected, there does not seem much cause for anxiety on this account on the tensile side, since the initial compression in the reinforcement increases its resistance to tension.

In the same way, the initial tension in the concrete on the compression side is an advantage, and if there is a reinforcement on this side the compressive stresses induced in it, although they add to the stresses due to loading, will not cause the metal to be stressed to anything approaching its safe resistance, as the compressive stresses due to loading cannot reach more than that in the concrete multiplied by the ratio of the moduli of elasticity of the two materials or about fifteen.

The initial stresses due to setting in air will, therefore, if anything, add to the resistance of the structure; the only result to be in any way feared is a slight cracking of the concrete on the tensile side, as the initial tensile stress will increase the deformation due to the loading.

A piece which is kept wet during the period of hardening will be affected in the opposite way, but the induced stresses are not so high as those which are induced by setting in air, and as they will not be very great they may safely be ignored.

With regard to temperature stresses; a fall in temperature will induce tensile stress in the concrete if held at the ends; it is very seldom, however, that the tensile side of a piece is subjected to much variation in temperature, it is usually the upper side which is exposed, as in the case of roofs or platforms of bridges, and it is only over the supports that there will be tensile stresses on the upper surfaces due to loading.

It is advisable to place special light reinforcements to resist the tensile stresses induced by a fall of temperature when the surfaces are exposed.

The compressive stresses due to a rise in temperature can safely be left to the concrete, as, when these occur, even if the structure is fully loaded, the concrete is perfectly able to resist them.

There is no doubt that there will be permanent deformations in reinforced concrete structures even under the application of small loads.

The effect of these has been referred to already, when it was shown that, after loading, the concrete on the compression side of a beam will tend to take a permanent set, but the metal in tension is perfectly elastic and would recover its original length if it was not held back by the concrete; as a consequence, the drag of the concrete on the reinforcement induces compressive stress, while the pull of the reinforcement on the concrete which has taken a permanent set induces tensile stresses.

It is therefore evident that the effect of the initial loading, if not applied too soon after moulding, is in reality to induce tensile stress in the concrete and compressive stress in the reinforcements, and since the concrete resists compressive stresses and the reinforcement tensile stresses, the structure is consequently stronger after initial loading, for the reason that the initial stresses in each material must be reduced to *nil* before any real resistance is required of them.

On the whole, it would seem that although initial stresses will certainly exist from several causes, it is not necessary to take them into consideration when designing a structure or to specially take them into account when assigning values to the safe working resistances.

That the resistance of the concrete in compression in pieces subjected to flexure varies uniformly from nil at the neutral axis to a maximum at the outer fibre.

This hypothesis has been fully discussed when dealing with the behaviour of reinforced concrete when subjected to flexure, and it has been shown that although the stress-strain curve for concrete is parabolic when extended to the ultimate resistance, it is sufficiently accurate to assume a straight line stress-strain relation when the calculations are based on the safe working resistance of the concrete.

The permission of this hypothesis includes the assumption of a constant modulus of elasticity for the concrete.

That the ratio of the moduli of elasticity of the steel and concrete is fifteen.

This hypothesis has also been fully dealt with when discussing the behaviour of reinforced concrete under direct compression.

To recapitulate. It has been shown that the following assumptions may be allowed when deducing the necessary formulæ for the calculation of reinforced concrete structures if these are based on the safe working stresses.

1. That the applied forces are perpendicular to the neutral axis of pieces subjected to bending.

2. That each fibre of the concrete acts by itself, not being affected by the contiguous fibres.

3. That there is always a solid contact between the reinforcement and the surrounding concrete.

4. That plane sections remain true planes during loading.

5. That there are no initial stresses on the structure to be designed.

6. That the resistance of the concrete in compression, in pieces subjected to flexure, varies uniformly from *nil* at the neutral axis to a maximum at the outer fibre, and that the modulus of elasticity of concrete is constant up to the limit of the safe working resistance.

7. That the ratio of the modulus of elasticity of steel to that of concrete is fifteen.

Safe Tensile Resistance of Steel.

The safe tensile resistance of steel may be assumed as one-half its elastic limit. For commercial bars of mild steel the elastic limit is generally from 32,000 to 34,000 pounds per square inch; the safe tensile resistance for this class of steel will therefore be from 16,000 to 17,000 pounds per square inch.

CHAPTER IV

METHODS OF CALCULATION

LIST OF SYMBOLS USED IN THE CALCULATIONS.

W = concentrated load.
w = distributed load per lineal unit.
P = pressure or load.
p = pressure per square unit.
K = shearing force.
k = shearing stress per square unit.
c = maximum compressive stress per square unit on the concrete subjected to direct stress or simple bending.
c_1 = minimum compressive stress per square unit on the concrete in pieces subjected to direct stress and bending combined.
c_2 = maximum compressive stress per square unit on the concrete in pieces subjected to direct stress and bending combined.
t = maximum tensile stress per square unit on concrete.
f = stress per square unit on the metal.
f_c = stress per square unit on the compressive reinforcement.
f_t = stress per square unit on the tensile reinforcement.
M = bending moment = moment of resistance.
T = Tangential stress, *i.e.* thrust normal to the radius in arches hoop tension or hoop compression.
H = Horizontal thrust in pieces subjected to direct stress and bending combined.
H = head of water.
R = Reaction at the springings of an arch.
R_S = ,, ,, ,, ,,
R_A = ,, ,, ,, ,,
bh = sectional area of the whole effective piece.
ω = sectional area of the metal.
ω_c = sectional area of the compressive reinforcement.
ω = sectional area of the tensile reinforcement.

a_s = sectional area of traverse shearing reinforcement.
d = total depth of a piece subjected to bending.
D = depth of the floor slab in ⊤ beams.
b = breadth of a rectangular piece subjected to bending or of the rib of a ⊤ beam.
B = breadth of the floor slab in a ⊤ beam.
R & r = radius.
δ = diameter.
o = perimeter of reinforcement.
L = span.
l = any length.
v = rise of arch or dome.
u = distance of the neutral axis in any section from the surface subjected to the greatest compressive stress.
a = distance of axis of the compressive reinforcement from the surface of the piece under greatest compressive stress.
β = distance of axis of the tensile reinforcement from the surface of the piece under greatest tensile or least compressive stress.
h = the maximum distance from the centre of gravity of any tensile reinforcement to that surface of any piece subjected to the greatest compressive stress.
$(u\text{-}a)$ = distance from the centre of gravity of the compressive reinforcement to the neutral axis.
$(h\text{-}u)$ = distance from the centre of gravity of the tensile reinforcement to the neutral axis.
$(u\text{-}a)_{max}$ = distance from the extreme edge of the compressive reinforcement, when it is of large sectional area, to the neutral axis.
$(h\text{-}u)_{max}$ = distance from the extreme edge of the tensile reinforcement, when it is of large sectional area, to the neutral axis.
ν = distance of the centre of gravity of symmetrical reinforcements from the *neutral surface* of a piece subjected to direct stress and bending combined.
ν_c = distance of the centre of gravity of the reinforcement, acting under the greatest compressive stress, to the *neutral surface* of a piece subjected to direct stress and bending combined.
ν_t = distance of the centre of gravity of the reinforcement acting under tension of the lesser compressive stress, to the

neutral surface of a piece subjected to direct stress and bending combined.

$v_{c\,max}$ = distance of the extreme edge of the reinforcement, acting under the greatest compressive stress, from the *neutral surface* of a piece subjected to direct stress and bending combined, when the reinforcement is of large sectional area.

$v_{t\,max}$ = distance of the extreme edge of the reinforcement, acting under tension of the lesser compressive stress, from the neutral surface of a piece subjected to direct stress and bending combined, when the reinforcement is of large sectional area.

E_c = coefficient of elasticity of concrete in compression.

E_f = coefficient of elasticity of the metal.

$m = \frac{E_f}{E_c}$

I = moment of inertia.

i_c = moment of inertia of compressive reinforcement about its own axis.

i_t = moment of inertia of tensile reinforcement about its own axis.

θ = angular circular measure.

A = angle in degrees.

ϕ = angle of inclined reinforcements to the horizontal.

$\tau = \frac{T}{bd}$

ϵ = elongation.

t° = degrees of temperature.

x, y = co-ordinates.

Σ = sign of summation.

π = 3·1416.

Bending Moments, Etc.

The weight of reinforced concrete may be taken as 150 lbs. per cubic foot.

The span of a beam or slab should in all cases be taken as extending from centre to centre of the supports.

There is no need to go very fully into the question of bending moments on beams and slabs, as these are fully dealt with in several well-known text books and pocket books, but it may be desirable to explain briefly the methods for dealing with pieces partially fixed at the supports, as is the case with most reinforced concrete floors.

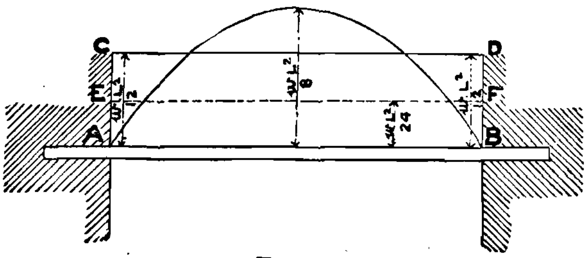


FIG. 14.

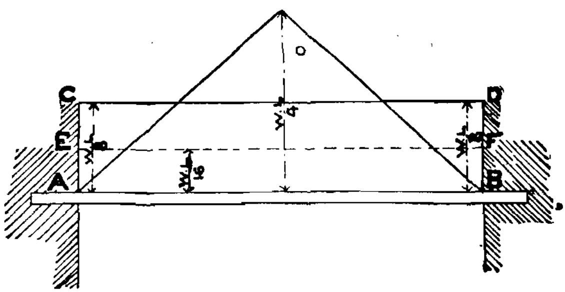


FIG. 15.

In Figs. 14 and 15, the parabola and triangle represent the bending moments on a freely supported beam, under a uniformly distributed and a central load respectively, giving bending moments at the centre of the span of

$$\frac{w\,L^2}{8} \text{ and } \frac{W\,L}{4},$$

where w is the distributed load per unit length and W is the concentrated load.

When the beams are absolutely fixed at the ends the closing line of the bending moment diagram is C D, which gives reverse bending moments at the supports of

$$-\frac{w\,L^2}{12} \text{ and } -\frac{W\,L}{8}$$

respectively. The bending moments at the centres of the spans must therefore be, in the first case,

$$\frac{w\,L^2}{8} - \frac{w\,L^2}{12} = \frac{w\,L^2}{24},$$

and in the second case,

$$\frac{W\,L}{4} - \frac{W\,L}{8} = \frac{W\,L}{8}.$$

When the ends of the beam are partially fixed the closing line will be horizontal if the fixing is the same at both ends or inclined if it is different at each end, but it must lie somewhere between the lines C D and A B.

The effect of the fixing must therefore be left to the judgment of the designer, as it is impossible to give values of the reverse bending moments at the supports, which will be absolutely accurate in all cases.

The assumption is generally made, however, of a value for the bending moment at the centre of a reinforced concrete beam or slab, which is supposed to take into account the fixing of the ends.

A very usual assumption for a uniformly distributed load is $\frac{w\,L^2}{10}$, giving for the reverse bending moments at the supports, if the fixing is assumed as being the same at both ends, of $-\frac{w\,L^2}{8} + \frac{w\,L^2}{10} = -\frac{w\,L^2}{40}$.

This is much too small an amount to allow, and if the reinforcement over the supports is calculated on this assumption it is extremely probable that the concrete will crack.

A better assumption to make is that the closing line of the diagram of bending moments lies halfway between C D and A B or at E F. This assumption gives a bending moment at the centre of the span of $\frac{w\ L^2}{12}$, and bending moments at the supports of

$$-\frac{w\ L^2}{8}+\frac{w\ L^2}{12}=-\frac{w\ L^2}{24},$$

and similarly for a central load the bending moment at the centre becomes $\frac{3\ W\ L}{16}$, and those at the supports $-\frac{W\ L}{16}$.

For practical purposes it is frequently advisable to assume the above values for the centre of the span, which will provide for partial fixing at the ends, and also to allow at the supports for absolute fixing, giving bending moments of $-\frac{w\ L}{12}$ in the first case and $-\frac{W\ L}{8}$ in the second.

With such allowances, we provide for either absolute or partial fixing, and although it is probable that the fixing is never absolutely perfect, and consequently the member will have an excess of reinforcement over the supports, we have the comfort of knowing that the error is on the side of safety.

These matters, however, as before pointed out, must be left to the judgment of the designer in all cases arising in practice.

When the piece extends over several spans and is supported on columns or walls the fixing is more likely to be perfect over the intermediate supports, and it is theoretically advisable to use the bending moments given for continuous beams for the intermediate supports and spans, while the outer spans may be calculated as for isolated spans on the assumptions previously explained. If the

TABLE IV.

Number of spans										
I	0	0								
II	0	$\frac{1}{8}$	0							
III	0	$\frac{1}{10}$	$\frac{1}{10}$	0						
IV	0	$\frac{3}{28}$	$\frac{2}{28}$	$\frac{3}{28}$	0					
V	0	$\frac{4}{38}$	$\frac{3}{38}$	$\frac{3}{38}$	$\frac{4}{38}$	0				
VI	0	$\frac{11}{104}$	$\frac{8}{104}$	$\frac{9}{104}$	$\frac{8}{104}$	$\frac{11}{104}$	0			
VII	0	$\frac{15}{142}$	$\frac{11}{142}$	$\frac{12}{142}$	$\frac{12}{142}$	$\frac{11}{142}$	$\frac{15}{142}$	0		
VIII	0	$\frac{41}{388}$	$\frac{30}{388}$	$\frac{33}{388}$	$\frac{32}{388}$	$\frac{33}{388}$	$\frac{30}{388}$	$\frac{41}{388}$	0	
IX	0	$\frac{56}{530}$	$\frac{41}{530}$	$\frac{45}{530}$	$\frac{44}{530}$	$\frac{44}{530}$	$\frac{45}{530}$	$\frac{41}{530}$	$\frac{56}{530}$	0

piece is supported on beams these will have a tendency to deflect, and it is probably best to calculate all the spans as if they were independent on the assumptions previously mentioned.

Tables IV. and V. give coefficients for multiplying $w\mathrm{L}^2$ to obtain the bending moments at the supports and on the spans respectively for pieces passing over several supports, and Table VI. gives the coefficients for multiplying $w\mathrm{L}$ to obtain the reactions at the supports.

TABLE V.

Number of spans									
I	0·125								
II	0·0703	0·0703							
III	0·08	0·025	0·08						
IV	0·0772	0·0364	0·0364	0·0772					
V	0·0775	0·0329	0·0461	0·0329	0·0775				
VI	0·0777	0·0340	0·0434	0·0434	0·0340	0·0777			
VII	0·0777	0·0334	0·044	0·0405	0·044	0·0334	0·0777		
VIII	0·0777	0·0339	0·0438	0·0412	0·0412	0·0438	0·0339	0·0777	
IX	0·0777	0·0363	0·0438	0·0410	0·042	0·0410	0·0438	0·0333	0·0777

In any case, for practical reasons, it is advisable to calculate all the spans for the worst conditions existing for any one span, since it is very uneconomical to construct

TABLE VI.

Number of spans									
I	1/2	1/2							
II	3/8	10/8	3/8						
III	4/10	11/10	11/10	4/10					
IV	11/28	32/28	26/28	32/28	11/28				
V	15/38	43/38	37/38	37/38	43/38	15/38			
VI	41/104	118/104	100/104	106/104	100/104	118/104	41/104		
VII	56/142	161/142	137/142	143/142	143/142	137/142	161/142	56/142	
VIII	153/388	440/388	374/388	392/388	386/388	392/388	374/388	440/388	153/388

each span of different dimensions and reinforcement as would be the case for a piece extending over several supports if different values were taken for the bending moments on each span. It may be possible to keep the concrete to the same dimensions and vary the reinforcements, but it is doubtful whether the additional labour spent on the calculations is justifiable.

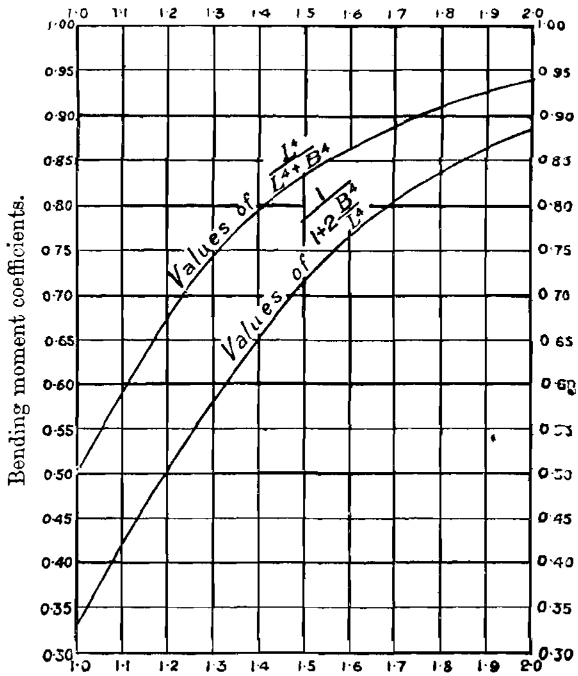
Slabs.

With a slab either built in or freely supported at the four edges, supposing (B) the smaller span and (L) the longer, according to the well-known formulæ of Grashof and Rankine, the bending moments given for beams must be multiplied by $\frac{L^4}{L^4 + B^4}$; where (B) is the span of the beam, and when (L) is the span of the beam the coefficient becomes $\frac{B^4}{B^4 + L^4}$. These coefficients cannot be absolutely correct, since they are derived from the assumption that each pair of strips, parallel to the longer and shorter sides respectively, act by themselves as if free from the adjacent strips. The problem of the distribution of the bending moments on slabs appears to be indeterminate; but the French Government Commission on Reinforced Concrete, in their report, give the values of the coefficients for the shorter and longer span as $\frac{1}{1 + 2\frac{B^4}{L^4}}$ and $\frac{1}{1 + 2\frac{L^4}{B^4}}$ respectively. These certainly appear to be more nearly the correct values than the coefficients given by Grashof and Rankine.

Table VII. gives the values of the respective coefficients for various ratios of L to B, and in Figs. 16 and 17 curves have been plotted giving the values from $\frac{L}{B} = 1$ to $\frac{L}{B} = 2$.

TABLE VII.

$\frac{L}{B}$	$\frac{L^4}{L^4 + B^4}$	$\frac{1}{1 + 2\frac{B^4}{L^4}}$	$\frac{B^4}{L^4 + B^4}$	$\frac{1}{1 + 2\frac{L^4}{B^4}}$
1	0·5	0·33	0·50	0·33
1·5	0·83	0·71	0·16	0·09
2	0·94	0·89	0·05	0·03

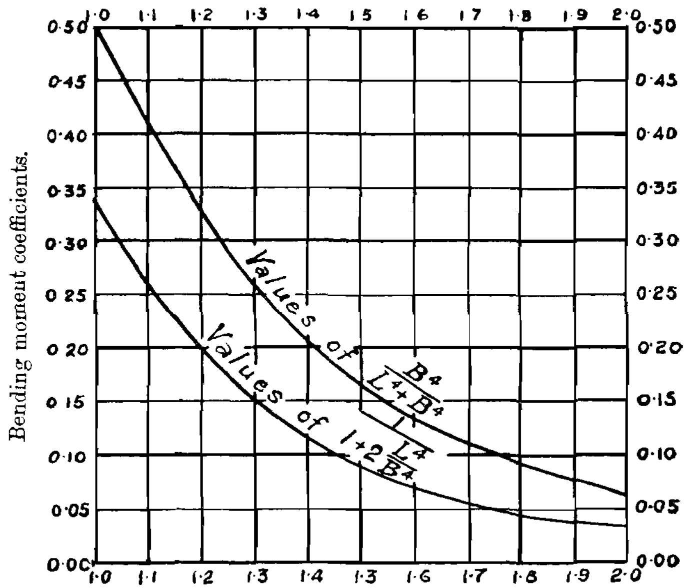


Values of $\frac{L}{B}$

FIG. 16.

For the purpose of calculating the bending moments on slabs, it is advisable to use the coefficients given by the French Government Commission.

For ratios of L to B greater than 2, it is only necessary



Values of $\frac{L}{B}$

FIG. 17.

to consider the lesser span, as no reinforcement will be necessary in the longitudinal direction.

The greatest bending moment will be that for the *shorter span*, but when reinforced concrete is under consideration, the stability parallel to the longer side must be provided for with reinforcing bars, and the necessary sections for these bars must be obtained for both spans.

We have, then, for a uniformly distributed load on a built-in slab of one span—using the values of M_C and M_A for the bending moments at the centre of the span and at the end supports—

For the *shorter span*,

$$M_C = + \frac{wB^2}{12} \times \frac{1}{1 + 2\frac{B^4}{L^4}} \quad . \quad . \quad [1]$$

$$*M_A = - \frac{wB^2}{24} \times \frac{1}{1 + 2\frac{B^4}{L^4}} \quad . \quad . \quad [2]$$

and for the *longer span*—

$$M_C = + \frac{wL^2}{12} \times \frac{1}{1 + 2\frac{L^4}{B^4}} \quad . \quad . \quad [3]$$

$$*M_A = - \frac{wL^2}{24} \times \frac{1}{1 + 2\frac{L^4}{B^4}} \quad . \quad . \quad [4]$$

For a square slab.

$$M_C = + 0{\cdot}028wL^2 \quad . \quad . \quad . \quad [5]$$

$$*M_A = - 0{\cdot}014wL^2 \quad . \quad . \quad . \quad [6]$$

For freely supported slabs the bending moment at the centre will be—

For the *shorter span*,

$$M_C = \frac{wB^2}{8} \times \frac{1}{1 + 2\frac{B^4}{L^4}} \quad . \quad . \quad [7]$$

* *See* remarks on page 88 as to advisability of increasing the value for the bending moment over the end support.

and for the *longer span*—

$$M_C = \frac{wL^2}{8} \times \frac{1}{1 + 2\frac{L^4}{B^4}} \quad . \quad . \quad [8]$$

When the slab passes over several supports, the values given in Tables IV. and V. may be used, remembering that for the *shorter span* the tabular coefficients must be multiplied by wB^2 and $\frac{1}{1 + 2\frac{B^4}{L^4}}$, and for the *longer span* by wL^2 and $\frac{1}{1 + 2\frac{L^4}{B^4}}$.

When using these values, however, it must be remembered that the slab is not absolutely fixed at the supports by reason of the deflexion of the beams, but this deflexion is so slight that for practical purposes we may neglect it.

The shearing forces may without much error be deduced by allowing the same coefficients of reduction as used for the bending moments.

For a uniformly distributed load the shearing forces will have the following values close to the support at the middle of the longer side—

$$K_B = \frac{1}{2}wB \times \frac{1}{1 + 2\frac{B^4}{L^4}} \quad . \quad . \quad [9]$$

and at the middle of the shorter support—

$$K_L = \frac{1}{2}wL \times \frac{1}{1 + 2\frac{L^4}{B^4}} \quad . \quad . \quad [10]$$

For a square slab—

$$K = \frac{1}{6}wL \quad . \quad . \quad . \quad . \quad [11]$$

Wind Pressures.

The wind pressure will vary somewhat according to the nature of the structure. It would be greater on a plain wall exposed on the front and rear faces than on a building with four sides, on account of the vacuum which is produced behind a thin structure by the force of the wind.

As a rule, a reinforced concrete structure is not of a thin nature, and we may consequently take from 30 to 40 lbs. per square foot as a maximum pressure in any but very exposed situations, or where the configuration of the land tends to concentrate the wind on the structure under consideration.

In such situations the limit should be increased to 50 or 55 lbs. per square foot.

When a building is of the usual kind it is generally unnecessary to inquire into its stability against wind pressure, as this will be amply provided for by the nature of the structure.

The walls on a line with the direction of the wind will in this case act as cantilevers, and the floors and roof as deep girders, connecting the exposed face to them in a perfectly rigid manner.

In the case of buildings with many windows or rectangular openings, it will be necessary to provide special reinforcements at the junctions of the vertical and horizontal framings, to strengthen these parts against any turning effect produced by the wind pressure on the exposed face. This is usually effected by placing inclined rods at the angles.

For isolated erections such as chimneys, telegraph and transmission line poles, towers, domes, spires and similar structures, the bending moment must be provided for.

The pressure of the wind is reduced when it acts on inclined surfaces; for a hemispherical surface the area of the vertical axial plane may be multiplied by 0·41; for a structure circular in plan the area on the vertical axial plane may be reduced by multiplying by 0·50; for an octagonal plan the multiplier will be 0·56, and for an hexagonal plan it may be taken as 0·66.

As a general rule, the centre of action of the wind pressure will be at the centre of gravity of the area of the plane on the vertical axis above the section under consideration.

In the case of telegraph and transmission line poles there will be two centres of pressure, one for the pole itself (the bending moment due to the force acting on this will be small), and the other that due to the force on the wires, which are generally assumed as being coated with ice.

CALCULATIONS.

Direct Compression.

When direct pressure is applied to a column, any transverse section is displaced parallel to itself, and if b and d are the lengths of the sides and ω is the area of the reinforcement, the area of the concrete will be $b\,d - \omega$.

The concrete and reinforcement are assumed to be deformed an equal amount, and the deformation causes unit stresses of c in the concrete and f in the reinforcement. The pressure (P) is therefore resisted by a combined stress $c\,(b\,d - \omega) + f\,\omega$ or $P = c\,(b\,d - \omega) + f\,\omega$ . . [1]

Now, if E_c is the modulus of elasticity of the concrete and E_f that of the reinforcement and both the concrete and metal are deformed an equal amount, we have the relation (as before explained) $c : E_c :: f : E_f$ or $f = c\,\frac{E_f}{E_c}$. [2]

For convenience of calculation the ratio $\frac{E_f}{E_c}$ is generally expressed as m. Therefore $f = m\,c$ and equation [1], giving the resistance of the piece, becomes

$$P = c\,\{b\,d + (m - 1)\,\omega\} \qquad . \quad . \quad [3]$$

We have allowed a value for m, or $\frac{E_f}{E_c}$, of 15.

Therefore we get $P = c\,\{b\,d + 14\,\omega\}$ . . . [4]

This equation may be used to find the load a piece will support by giving values to c and ω.

Now the pressure per square unit $= p = \frac{P}{bd}$.

We have allowed 500 lbs. per square inch for the value of c. Assuming any ratio ψ of ω to $b\,d$, ω becomes $\psi\,b\,d$, and substituting these values in equation [4], we get

$$p = 500\,(1 + 14\,\psi) \qquad . \quad . \quad . \quad . \quad . \quad [5]$$

From this equation values of p may be found for various ratios of reinforcement, and tabulated.

These values have only to be multiplied by the sectional area of the piece to give the loads which can be borne with any ratio of reinforcement. (*Vide* "Manual of Reinforced Concrete, etc.," p. 102.)

From such a table, a diagram can be plotted giving at a glance the sizes of columns and their reinforcements to support any loading. (*Vide* "Manual of Reinforced Concrete, etc.," p. 104.)

Columns with Eccentric Loads.

If the loading on a column is eccentric (*i.e.* is not applied at the centre), the effect is not altered if we imagine two additional forces equal to the loading in intensity and acting at the centre of the column in opposite directions parallel to the direction of the loading.

This is the same as substituting for the original loading a load of equal intensity acting at the centre of the column, and a couple acting with a lever arm equal to the distance of the load from the centre of the column. This gives us a direct thrust and a bending moment and we proceed by the use of the formulæ, pp. 185 to 199.

Hooped Columns.

The resistance of hooped columns is at present usually calculated in a more or less empirical manner. The French Government Commission on Reinforced Concrete, after fully considering the question, decided on a formula for obtaining the value of the safe compressive resistance to be used in the equation $P = c\,(0{\cdot}7854d^2 + 14\,\omega)$ . . . [4a] which is similar to equation [4]; d being the diameter of the hooped core and ω the sectional area of the longitudinal bars.

This formula may be stated for pitches of spirals between $\frac{1}{5}\,d$ and $\frac{1}{8}\,d$, where d is the diameter of the hooped core of concrete.

$$c = 0{\cdot}28\ C\ (1 + 32\ \psi_h) \qquad . \quad . \quad . \quad . \quad [6]$$

where C is the ultimate compressive resistance of the concrete at 90 days and ψ_h is the ratio of the volume of the hooping reinforcement to the volume of the concrete core within the hooping.

The resistance of various mixtures as given by the French Government are set out in Table VIII.

TABLE VIII.

Pounds of cement to $13\frac{1}{2}$ cubic feet of sand and 27 cubic feet of stone.	Ultimate resistance of concrete at 90 days.
853	2,844 lbs. per square inch.
718	2,560 ,, ,,
632	2,275 ,, ,,

The last of these is approximately a 1 : 2 : 4 concrete.

The Commission restricted the c given by equation [6] to a value not greater than 0·6 the ultimate resistance of the concrete.

M. Considère recommends that the percentage of area of the longitudinal reinforcement should not be less than 0·5 per cent. that of the concrete in the core, and their number not less than six.

The Austrian Government rules allow only the safe resistance of the concrete for direct compression, but the area of the piece used for the purpose of calculation may be increased to the amount given by the equation—

$$A = A_c + 15\ A_l + 30\ A_h, \qquad . \quad . \quad [7]$$

where A is the equivalent area, A_c is the area of the concrete within the hooping, A_l is the sectional area of the longitudinal reinforcement, and A_h is the sectional area of an imaginary longitudinal reinforcement, the weight, or volume, of which is equal to that of the hooping reinforcement, both weights, or volumes, being reckoned per unit length of the piece.

These methods of calculation give safe results and may be used with confidence, particularly the French rule, which limits the compressive resistance.

It would appear, however, that the most rational method to adopt is to find the area of the hooping which will safely resist the unit pressure, and then to add longitudinal reinforcements, to tie the hooping together, and to increase the resistance to the amount required.

To effect this it becomes necessary to know the ratio of the direct pressure to the lateral pressure.

This can be found from the results of actual experiments where failure has occurred by the breaking of the hooping reinforcement if the ultimate strength of the reinforcement is known.

In such a case we know the unit direct pressure p, and the unit hoop tension exerted, and can calculate the unit lateral pressure from the equation for hoop tension, or $T = \frac{q\,d}{2}$, where T is the unit hoop tension and q is the unit lateral pressure. From this equation we get $q = \frac{2\,T}{d}$.

It appears from the results of experiments that a safe ratio of $\frac{p}{q} = 10$.

Adopting a working resistance of 1,100 lbs. per square inch in the longitudinal direction for hooped pieces of 1 : 2 : 4 concrete, or a mixture having 630 lbs. of cement to 13½ cubic feet of sand and 27 cubic feet of stone, we get $\frac{p}{q} = \frac{1{,}100}{q} = 10$, or $q = 110$ lbs. per square inch.

And as $T = \frac{q\,d}{2}$, $T = 55\,d$ per inch width.

If the hooping is of mild steel the safe resistance in tension may be assumed as 16,000 lbs. per square inch, consequently the area of spiral per inch width will be

$$\frac{55\ d}{16{,}000} = \frac{d}{290} \text{ square inches.}$$

If the pitch of the hooping is $\frac{1}{5}\ d$, then the area of the spiral will be $\frac{d}{290} \times \frac{d}{5} = \frac{d^2}{1{,}450}$, giving a diameter for the hooping, if round—

$$\delta = \sqrt{\frac{d^2}{1{,}450 \times 0{\cdot}7854}} = 0{\cdot}029\ d. \quad . \quad [8]$$

Similarly, for a pitch of $\frac{1}{6}\ d$, the area of the spiral will be $\frac{d^2}{1{,}740}$, and its diameter, if round, will be

$$\delta = \sqrt{\frac{d^2}{1{,}740 \times 0{\cdot}7854}} = 0{\cdot}0275\ d\ ; \quad . \quad [9]$$

for a pitch of $\frac{1}{7}\ d$ the area of the spiral will be $\frac{d^2}{2{,}030}$, and its diameter, if round, will be

$$\delta = \sqrt{\frac{d^2}{2{,}030 \times 0{\cdot}7854}} = 0{\cdot}025\ d\ ; \quad . \quad [10]$$

and for a pitch of $\frac{1}{8}\ d$ the area of the spiral will be $\frac{d^2}{2{,}320}$, and its diameter, if round, will be

$$\delta = \sqrt{\frac{d^2}{2{,}320 \times 0{\cdot}7854}} = 0{\cdot}0234\ d. \quad . \quad [11]$$

In hooped members the spirals or hoopings are spaced close together and consequently the longitudinal rods are not necessary as distribution rods; they are useful, however, for the practical purpose of tying the hooping together, and they will also add considerably to the resistance of the member.

If the spiral cannot be in a continuous length, the several lengths must overlap for two spirals, and their ends be bent into the centre of the core.

It is usual to employ from six to eight longitudinal rods. With hoopings of the sizes previously found, the unit compressive resistance of the concrete is 1,100 lbs. per square inch, and consequently the unit resistance afforded by the longitudinal reinforcements will be 1,100 × 15 = 16,500 lbs. per square inch, and if ω is their combined area, the total resistance of the member becomes

$$P = 1{,}100 \left\{ \frac{\pi d^2}{4} - \omega + 15\,\omega \right\}$$

or $$P = 1{,}100 \left\{ 0{\cdot}7854\, d^2 + 14\,\omega \right\} \qquad [12]$$

where d is the diameter of the hooped core in inches.

Longitudinal Direct Stresses in Pieces Subjected to Flexure.

Single Reinforcements.

If the reinforcements are of small sectional area, as is usually the case, we may consider the stresses in the metal as of uniform intensity over the whole area, and that they act at the centre of gravity or axis of the section.

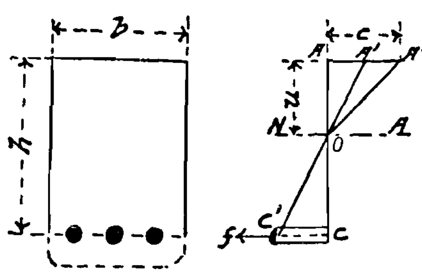


FIG. 18.

The compressive resistance in this case is that due to the concrete on the compressive side of the neutral axis, and the tensile resistance is supposed to be supplied by the reinforcement only. The compressive resistance is therefore represented by the triangle A O H″ × b (Fig. 18), and since A A″ represents the maximum compressive resistance of the concrete, the above expression becomes $\frac{1}{2}\ c\ u\ b$ if c is the maximum resistance of the concrete in compression and u the distance from the outer compressive fibre of the concrete to the neutral axis.

The tensile resistance is represented by ωf, where ω is the combined area of the reinforcements and f the resistance of the metal.

Since the compressive and tensile resistances must be equal, we have $\frac{1}{2}\ c\ u\ b = \omega f$ . . . . . . [13]

From the hypotheses of the conservation of plane sections

$$\text{A A}' : \text{C C}' :: \text{O A} : \text{O C}.$$

But $\quad \text{A A}' \times \text{E}_c = c, \text{ or A A}' = \frac{c}{\text{E}_c}$

and $\quad \text{C C}' \times \text{E}_f = f, \text{ or C C}' = \frac{f}{\text{E}_f}$

also $\quad \text{O A} = u \text{ and O C} = (h - u)$

$$\therefore \frac{c}{\text{E}_c} : \frac{f}{\text{E}_f} :: u : (h - u), \text{ or } \frac{c\ (h - u)}{\text{E}_c} = \frac{f\ u}{\text{E}_f}$$

$$\therefore f = \frac{\text{E}_f}{\text{E}_c}\ c\ \frac{(h - u)}{u}, \text{ but } \frac{\text{E}_f}{\text{E}_c} = m = 15$$

$$\therefore f = m\ c\ \frac{(h - u)}{u} \qquad . \quad . \quad . \quad . \quad . \quad . \quad [14]$$

Now substituting this value of f in equation [13], we get

$$\tfrac{1}{2}\ u^2 b - m\ \omega\ (h - u) = 0;$$

from which

$$u = \frac{m\ \omega}{b}\left[\sqrt{1 + 2\ \frac{b\ h}{m\ \omega}} - 1\right] \qquad . \quad . \quad [15]$$

Since the stress-strain curve of the concrete in compression is a straight line, the centre of action of the compressive stresses is at a plane one-third of the height of O A or u from A, and the lever arm of the couple of resisting forces will be $\left(h - \frac{u}{3}\right)$.

And since the resisting moment must equal the bending moment,

$$M = \frac{c\,u\,b}{2}\left(h - \frac{u}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad [16]$$

similarly,

$$M = f\,\omega\left(h - \frac{u}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad [17]$$

These are the fundamental equations for a single reinforcement.*

Double Reinforcements.

The reinforcements are supposed of small sectional area as in the case of single reinforcements. In this instance the compressive resistance is that of the concrete on the compression side of the neutral axis added to that of the compressive reinforcement, and the tensile resistance is supplied by the reinforcement on the tensile side of the neutral axis.

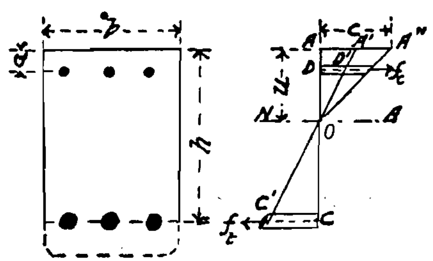


FIG. 19.

The compressive resistance is therefore represented by the area of the triangle A O A″ × b (Fig. 19) + $\omega_c \times f_c$.

* For methods of employing these equations and tables and diagrams based upon them, *vide* "Manual of Reinforced Concrete, etc."

Since A A″ represents the maximum compressive resistance of the concrete and the area A O A″ equals one-half the surrounding rectangle, this expression becomes

$$\tfrac{1}{2}\, c\, u\, b + \omega_c\, f_c.$$

The tensile resistance is represented by $\omega_t\, f_t$.

And as the compressive and tensile resistances must be equal, we have

$$\tfrac{1}{2}\, c\, u\, b + \omega_c\, f_c = \omega_t\, f_t \quad . \quad . \quad . \quad . \quad [18]$$

From the hypothesis of the conservation of plane sections we have

$$\text{A A}' : \text{C C}' : \text{D D}' :: \text{O A} : \text{O C} : \text{O D}.$$

But A A′ × $E_c = c$ or A A′ $= \dfrac{c}{E_c}$; and, similarly, C C′ $= \dfrac{f_t}{E_f}$ and D D′ $= \dfrac{f_c}{E_f}$.

Also O A $= u$, O C $= (h - u)$ and O D $= (u - a)$.

We get, therefore, $\dfrac{c}{E_c} : \dfrac{f_t}{E_f} : \dfrac{f_c}{E_f} :: u : (h - u) : (u - a)$;

consequently $\dfrac{c\,(h - u)}{E_c} = \dfrac{f_t\, u}{E_f}$ or $f_t = c\,\dfrac{E_f}{E_c}\,\dfrac{(h - u)}{u}$

$$\therefore f = c\, m\, \frac{(h - u)}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad [19]$$

$$\text{and } \frac{c\,(u - a)}{E_c} = \frac{f_c\, u}{E_f} \text{ or } f_c = c\, m\, \frac{(u - a)}{u}\,. \quad . \quad . \quad [20]$$

$$\text{and } \frac{f_t\,(u - a)}{E_f} = \frac{f_c\,(h - u)}{E_f} \text{ or } f_c = f_t\,\frac{(u - a)}{(h - u)}\,. \quad . \quad [21]$$

Also, since the resisting moment must equal the bending moment,

$$M = \tfrac{1}{2}\, c\, u\, b \times \tfrac{2}{3}\, u + (u - a)\,\omega_c\, f_c + (h - u)\,\omega_t\, f_t$$

$$\text{or } M = \tfrac{1}{3}\, c\, u^2 b + (u - a)\,\omega_c\, f_c + (h - u)\,\omega_t\, f_t\,. \quad [22]$$

Substituting f_t and f_c from equations [19] and [20] in equations [18] and [22], we have

$$\tfrac{1}{2} u^2 b + m \{\omega_c (u - a) - \omega_t (h - u)\} = 0$$

$$\text{or } u = \frac{m}{b}(\omega_c + \omega_t)\left[\sqrt{1 + \frac{2b\,(a\,\omega_c + h\,\omega_t)}{m\,(\omega_c + \omega_t)^2}} - 1\right]. \quad [23]$$

$$\text{and } \mathrm{M} = \frac{c}{u}\left[\tfrac{1}{3} u^3 b + m \{\omega_c (u - a)^2 + \omega_t (h - u)^2\}\right] \quad [24]$$

As before, m will be 15.

These are the fundamental equations for double reinforcements.* For designing structures it is convenient to give (a) the value of $\frac{1}{3} u$, thus placing the compressive reinforcement at the centre of the compressive resistance of the concrete. We can then use the equation

$$\mathrm{M} = \omega_t f_t \left(h - \frac{u}{3}\right), \quad . \quad . \quad . \quad [25]$$

and if m is given its value of 15, the equation for u becomes

$$u = \frac{5}{b}(2\,\omega_c + 3\,\omega_t)\left[\sqrt{1 + \frac{b}{5}\left\{\frac{h\,\omega_t}{(2\,\omega_c + 3\,\omega_t)^2}\right\}} - 1\right] \quad [26]$$

Generally speaking, a double reinforcement is not economical when used in beams or slabs unless high percentages of reinforcement are necessary, which is seldom the case. It frequently happens, however, that the available depth is restricted, in which case it becomes necessary to use a compressive reinforcement to obtain the requisite resistance.

Some people, when using a double reinforcement, the bars being placed symmetrically on each side of the neutral axis, calculate the necessary reinforcement of a beam as if it were a steel girder, neglecting the resistance of the

* For methods of employing these equations, and tables and diagrams based upon them, *vide* "Manual of Reinforced Concrete, etc."

concrete. This method is wrong, as the concrete must in reality support some of the stresses, however the reinforcement may have been calculated, and the assumption that the reinforcement will take the entire stress cannot be true.

If the compressive reinforcement is calculated to offer a high resistance, the concrete at the outer fibre must be stressed more than one-fifteenth of this amount if $\frac{E'}{E_c} = 15$, or if the reinforcement is stressed to, say, 16,000 lbs. per square inch, the concrete at its axis is stressed to 1,067 lbs. per square inch, and if the reinforcement is at the centre of the compressive resistance of the concrete, the outer fibre must be stressed to 1,600 lbs. per square inch, or 2·67 times the permissible safe resistance. As a matter of fact, when a compressive reinforcement is used, the concrete in compression must be calculated as offering only the maximum allowed resistance at the outer fibre, and consequently the compressive reinforcement cannot offer a resistance of more than 6,000 lbs. per square inch if it is placed at the centre of the compressive resistance of the concrete.

When the bending may be on either side, as in division walls of reservoirs and like structures, a double reinforcement is, of course, essential.

Reinforcements are also necessary, extending into beams or slabs, where reverse bending moments will occur over the supports. For this purpose some of the tensile bars are frequently bent up at their ends, and thus afford resistance to the diagonal tensile stresses near the supports. In most cases where the tensile bars are bent up further reinforcements will be required, in addition, to resist the reverse bending moments, as the top bend of the tensile bars should be at the support if they are to resist the diagonal tensile stresses.

T Beams with Single Reinforcements.

In the case of a T-shaped beam, the floor slab which forms the top of the T will resist the compressive stresses.

It is necessary, therefore, to inquire into the width of the slab, which may be considered as acting with the T beam. It is obvious that if the slab has already been calculated as offering the maximum allowed compressive resistance when acting by itself we cannot consider it as again offering the maximum resistance when forming part of the T beam, unless the stresses are acting in a direction normal to those on the slab, and even in this case it is doubtful if we can allow for the resistance of the whole width of the slab. It is also evident that when the distance between the beams is great in proportion to the span the effect of the flexure of the beam will only extend for a part of the distance from the beam to the centre of the distance between the beams.

It is difficult to determine the exact width of slab on the side of the beam which may be allowed to act with it, and we must therefore assume some safe limits.

The committee appointed by the Royal Institute of British Architects to report on reinforced concrete, after careful consideration, recommended that the width of the slab to be considered as acting with the T beam might be assumed as one-third the span or three-fourths the distance between the centres of the beams, whichever is the smaller dimension. This applies to secondary beams where there are series of beams at right angles to one another, or to all the beams when they are only placed in one direction, since, in such cases, the maximum compressive stress in the upper portion of the slab, when considered by itself, will act in a direction normal to the beams, and consequently will not effect the compressive resistance of the slab in the direction of the beams. If the width of the panel is greater than half its length, there will be compressive stresses at the upper surface on a direction parallel with the beams, but, provided the ratio is not much greater than one-half, the effect of the loading on the slab in the direction of the beams will be small.

When the floor is divided into an equal number of bays, by secondary beams supported on main beams, the maximum compressive stress in the main beams will be over a secondary beam, and consequently these widths will also apply to the main beams in such a case.

When the bays of a floor are square, or nearly so, the floor slab, considered by itself, will have been designed with the maximum compressive stresses in a direction parallel to the beams, and in this case the width of slab acting with the beam should be reduced to one-half the distance between the centres of the beams, and the same applies to the main beams when the floor is divided into an uneven number of bays by secondary beams supported on main beams.

When the neutral axis of a T beam coincides with or is above the underside of the slab, the beam may be calculated as a rectangular beam of a width equal to the width of the slab acting with the T beam, and the equations given for rectangular beams will apply.

If, however, the neutral axis falls below the underside of the slab, we cannot employ these equations.

In such a case it is usual to neglect the resistance offered by the small portion of the rib above the neutral axis, as it simplifies the equations and the neglected resistance is small and tends to safety

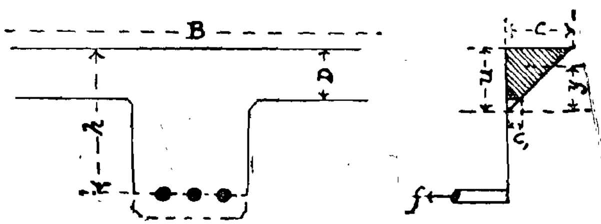


FIG. 20.

If we call the maximum compressive resistance c, and the resistance at the underside of the slab c_1, the depth of the slab D and the width B,

$$c_1 = c\,\frac{u - D}{u} \quad . \quad . \quad . \quad . \quad [27]$$

and as the compressive and tensile resistances must be equal,

$$\frac{c + c_1}{2} \times B\,D = \omega f \quad . \quad . \quad . \quad [28]$$

and as in the case of rectangular beams, from the hypothesis of the conservation of plane sections, we obtain the equation

$$f = c\,m\,\frac{h - u}{u} \quad . \quad . \quad . \quad . \quad [29]$$

Inserting the values of c and f from [27] and [28] in [29], we get

$$u = \frac{\dfrac{B\,D^2}{2} + m\,\omega\,h}{B\,D + m\,\omega} \quad . \quad . \quad . \quad [30]$$

The distance of the centre of gravity of **the trapezium** $c\ c_1$ from the upper surface is

$$u - y = \frac{D}{3} \frac{c + 2c_1}{c + c_1} \quad . \quad . \qquad [31]$$

For dividing the hatched portion in Fig. 20 into two triangles, and taking moments about the upper surface, we get

$$\frac{c}{2} \cdot \frac{D}{3} + \frac{c_1}{2} + \frac{2D}{3} = \frac{c + c_1}{2} (u - y)$$

or

$$\frac{c + 2c_1}{2} \cdot \frac{D}{3} = \frac{c + c_1}{2} (u - y)$$

$$\therefore (u - y) = \frac{D}{3} \frac{c + 2c_1}{c + c_1}.$$

Inserting the value of c_1 from [27], we get

$$u - y = \frac{3uD - 2D^2}{3(2u - D)},$$

multiplying the numerator and denominator of the right side of this equation by 2, we get

$$u - y = \frac{3D(2u - D) - D^2}{6(2u - D)} = \frac{D}{2} - \frac{D^2}{6(2u - D)},$$

or

$$y = u - \frac{D}{2} + \frac{D^2}{6(2u - D)} \quad . \quad . \quad . \quad . \quad [32]$$

Also,

$$M = (y + h - u) f \omega \quad . \quad . \quad . \quad [33]$$

and substituting the value of $f\ \omega$ from [28] and further substituting the value c_1 from [27], we get

$$M = \frac{cBD}{2u} (y + h - u)(2u - D)$$

and inserting the value of y from [32], we get

$$M = \frac{c\,B\,D}{6\,u}\,[D\,(2\,D - 3\,u) + 3\,h\,(2\,u - D)] \quad . \quad [34]$$

These are the fundamental equations for singly reinforced T beams.*

T BEAMS WITH DOUBLE REINFORCEMENT OF SMALL SECTIONAL AREA.

If the neutral axis is above or coincides with underside of the slab, all the equations for doubly reinforced rectangular beams will apply if the width B is substituted for b.

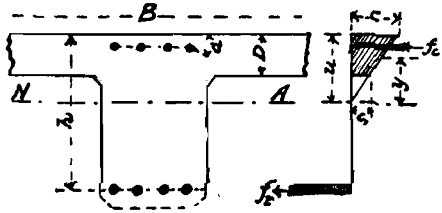


FIG. 21.

As in the case of singly reinforced T beams,

$$c_1 = c \times \frac{u - D}{u} \quad . \quad . \quad . \quad [35]$$

and from the hypothesis of the conservation of plane sections in a similar manner to doubly reinforced rectangular beams (Fig. 21),

$$f_t = m\,c\,\frac{h - u}{u} \quad . \quad . \quad . \quad . \quad [36]$$

$$f_c = m\,c\,\frac{u - a}{u} \quad . \quad . \quad . \quad [37]$$

and since the compressive and tensile resistances must be equal,

$$\frac{c + c_1}{2} \times B\,D + \omega_c\,f_c = \omega_t\,f_t \quad . \quad . \quad [38]$$

* For methods of employing these equations and diagrams based upon them, *vide* "Manual of Reinforced Concrete, etc."

Inserting c_1 from [35], f_t from [36], and f_c **from [37]** in equation [38], we get

$$u = \frac{\frac{B\ D^2}{2} + m\ (\omega_t\ h + \omega_c\ a)}{B\ D + m\ (\omega_t + \omega_c)} \quad . \quad . \quad [39]$$

As in the case of single reinforcements,

$$y = u - \frac{D}{2} + \frac{D^2}{6\ (2\ u - D)} \quad . \quad . \quad [40]$$

$$M = \frac{c + c_1}{2}\ B\ D\ y + \omega_c f_c\ (u - a) + \omega_t f_t\ (h - u)$$

Substituting the values of c_1, f_c and f_t from equations [35], [37], and [36], we get

$$M = \frac{c}{u}\left[y\ B\ D\ \frac{2\ u - D}{2} + m\left\{\omega_c (u - a)^2 + \omega_t\ (h - u)^2\right\}\right]$$

Inserting the value of y from equation [40], we get

$$M = \frac{c}{u}\left[B\ D\left(u^2 - u\ D + \frac{D^2}{3}\right) + m\left\{\omega_c\ (u - a)^2 + \omega_2\ (h - u)^2\right\}\right] \quad . \quad . \quad . \quad [41]$$

These are the fundamental equations for doubly reinforced **T** beams.

Inverted T Beams.

This form of beam applies to the ends of **T** beams when fixed or partially fixed.

Since the neutral axis in this case will always fall outside the slab, the calculations are made as for rectangular beams of the width of the rib.

The Arrangement of the Tensile Reinforcements in Beams.

In all beams it is advisable, if possible, to place the longitudinal tensile reinforcements in two layers, and to bend up the bars in the top layer towards the supports, as such bent up bars assist considerably in resisting the diagonal tensile stresses; the bars should be so bent up that they reach the centre of the compressive stresses at the support and are here bent over so as to extend over or into the supports. The remainder of the rods should be continued along near the tensile surface well over or into the supports.

In continuous beams any bent up bars should be extended near the upper surface, and bent down into the adjoining beam. Additional bars are generally necessary to resist the tensile stresses induced near the upper surface by the reverse bending moments over the supports.

If the bars are not of sufficient length to allow of their being carried through the adjoining beam, they should be extended near the upper surface for some distance in the beam, the reinforcements of the adjoining beam being also continued for some distance near the upper surface. Thus the bars of neighbouring beams pass one another over the support. In this case it is probable that no additional reinforcement will be required over the supports.

Another method of continuing the reinforcement from beam to beam is to allow the bars to overlap one another where inclined, and to bind them with annealed wire at this place. The intensity of tensile stress is very small through this portion of their length, and consequently it is a good place to form the overlap.

To avoid congestion of bars over the supports, it is frequently advisable not to extend the bars, bent up from the bottom, over the supports, but to hook these, at the edge of the support, over additional bars placed near the upper surface.

The lower bend of any bent up bars should not be farther from the supports than one-sixth the span in a freely supported beam, or one-third of the span when the beam is fixed or partially fixed at the ends or continuous.

It is advisable to extend the top reinforcements about one-eighth the span into the beam from the end supports and one-fourth the span on either side from the intermediate supports, and any additional reinforcements over supports should extend for like distances.

Some constructors use special forms of bar or a special arrangement of bars, in which case the above methods of arrangement will not apply, but in all cases it is necessary to provide bars to resist the tensile stresses over the supports.

Pieces with Reinforcements of Large Sectional Area (Fig. 22).

In this case the reinforcement has in itself a resistance against bending, and neither the depth nor the sectional area can be neglected when compared to that of the whole piece.

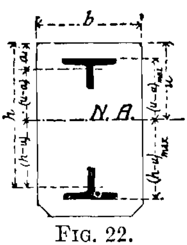


Fig. 22.

The values f_c and f_t will therefore be considered as the mean and f_{cmax} and f_{tmax} as the maximum resistances of the sections: $(u-a)$ and $(h-u)$ being the distances from the neutral axis to the centres of gravity of the metal sections; and $(u-a)_{max}$ and $(h-u)_{max}$ the distances from the neutral axis to the outer fibres of the reinforcements, and further i_c and i_t will be the moments of inertia of the reinforcements about their own centres of gravity.

The compressive reinforcement takes up an appreciable area compared with that of the whole piece, which must be deducted from the area of the concrete in compression. The resistance of the concrete displaced by this reinforcement may be supposed to act at its centre of gravity.

In this case the stress on the area of concrete replaced by the upper reinforcement will be—

$$c\frac{(u-a)}{u}.$$

Equating the compressive and tensile stresses we get—

$$\tfrac{1}{2}cub - \frac{(u-a)}{u}c\omega_c + \omega_c f_c = \omega_t f_t. \quad . \quad . \quad [42]$$

For the calculations of the moment of resistance of the piece, the moment of each reinforcement acting by itself must be included, and for the compressive reinforcement the moment of the concrete which it replaces must be deducted.

We have therefore—

$$M = \frac{1}{3}cu^2b - \frac{(u-a)^2c\omega_c}{u} - \frac{ci_c}{u} + f_c\omega_c(u-a) + \frac{f_ci_c}{(u-a)} +$$

$$f_t\omega_t(h-u) + \frac{f_ti}{(h-u)}. \quad . \quad . \quad . \quad [43]$$

We have also—

$$f_c = cm\frac{(u-a)}{u}, \quad . \quad . \quad . \quad [44]$$

$$f_t = cm\frac{(h-u)}{u}. \quad . \quad . \quad . \quad [45]$$

Substituting these values in equations [42] and [43], we get—

$$\frac{1}{2}cub - \frac{(u-a)}{u}c\omega_c + \omega_c cm\frac{(u-a)}{u} - \omega_t cm\frac{(h-u)}{u} = 0, \quad [46]$$

or—

$$\tfrac{1}{2}u^2b - (u-a)\omega_c(1-m) - m\omega_t(h-u) = 0, \quad . \quad [47]$$

and—

$$\mathbf{M} = \frac{1}{3}cu^2b - \frac{(u-a)^2c\omega_c}{u} - \frac{ci_c}{u} + cm\frac{(u-a)^2}{u}\omega_c$$

$$+ \frac{cmi}{u} + \frac{cm(h-u)^2}{u}\omega_t + \frac{cmi_t}{u}, \quad . \quad . \quad . \quad [48]$$

$$\mathbf{M} = \frac{c}{u}\left[\frac{1}{3}u^3b + \left\{(u-a)^2\omega_c + i^c\right\}(m-i)\right.$$

$$\left. + m(h-u)^2\omega_t + i_t\right]. \quad . \quad . \quad [49]$$

The maximum stresses in the reinforcements are given by—

$$f_{cmax} = cm\frac{(u-a)_{max}}{u}, \quad . \quad . \quad [50]$$

and $$f_{tmax} = cm\frac{(h-u)_{max}}{u}. \quad . \quad . \quad [51]$$

If there is only a reinforcement on the tensile side

$$f = cm\frac{(h-u)}{u}, \quad . \quad . \quad [52]$$

$$\tfrac{1}{2}u^2b - m(h-u)\omega = 0, \quad . \quad . \quad [53]$$

$$\mathbf{M} = \frac{c}{u}\left[\frac{1}{3}u^3b + m\{(h-u)^2\omega + i\}\right]. \quad [54]$$

$$f_{max} = cm\frac{(h-u)_{max}}{u} \quad . \quad . \quad [55]$$

The equations for T beams reinforced with large sections follow from the above, being compiled in exactly the same manner.

The employment of reinforcements of large sectional area is not to be recommended, as the best practice in reinforced concrete construction for pieces subjected to simple bending is to keep the metal as far as possible from the neutral axis of the piece, where it acts at its best advantage. The design of reinforced concrete is similar to that of metal girders in this respect.

The rolled joist is uneconomical as regards the use of the metal, its economy being a purely practical consideration. The same remark applies to a plate-web girder, true economy of material being only obtained when the web is of the open type, being only sufficient to take up the shearing stresses. With reinforced concrete the practical economy of the rolled joist and plate web does not apply, as the cost of construction is very nearly the same whether we use large sections or small.

In the case of slabs, the tests and experiments which have been carried out prove that the best type is that when small sections of reinforcement are employed fairly close together; and the fact that but few constructors employ large reinforcing sections in beams is a sufficient proof that small sections are the most economical for these.

For arches which act mainly in compression, large sections may be employed with economy, and many constructors, Melan and Wünsch among the number, employ this type with success. For arches large sections have also the advantage of readily adapting themselves to hinged connexions.

When large reinforcing sections are employed it is the usual practice to use them for the purpose of supporting or partially supporting the falsework, and in consequence they gain an extensive practical advantage; but it cannot be considered good practice to support the falsework by the aid of the reinforcement, as, when this is done, there must be a certain amount of initial strain and there is more risk of vibration. These remarks do not, of course, refer to the use of rolled joists as beams, this form of construction not being reinforced concrete in the true sense of the term.

Bond and Shearing Stresses.

The vertical shear on a piece under flexure is resisted by the longitudinal shearing resistance of the fibres.

The longitudinal or horizontal shearing resistances add the increments to the direct resistance of the fibres. If there were no shearing resistances between the fibres they would simply slide on one another and exert no resistance to bending.

On each small square particle of a beam there are horizontal and vertical forces acting; the vertical forces being the direct shear and the horizontal forces the resistance of the fibres. These must be equal to one another if equilibrium is to be maintained.

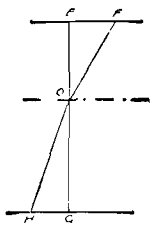


Fig. 23.

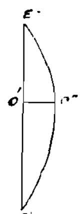


Fig. 24.

The shearing resistances that it is necessary to assure in a beam are in reality those acting horizontally, which counteract the tendency for the fibres to slide over one another, and thus prevent the failure of the beam.

Now it is clear that, the added increments of the longitudinal stresses in a homogeneous beam being represented by the triangles O E F and O G H (Fig. 23), the longitudinal shearing stress on any horizontal plane must be equal to the sum of the increments of the longitudinal stresses above that plane. The effect of this is shown graphically in Fig. 24, where E′ O′ G′ represents the vertical plane section, upon which are acting shearing stresses at each point in its height, these shearing stresses being represented by the ordinates from E′ O′ G′ to the curve E′ O″ G′. It will be seen that these increase from the top surface until the neutral axis is reached. Below the neutral axis the direction of the action of the stresses becomes the opposite to that above, and consequently the increments of the shearing resistances become *minus* quantities and the curve bends towards E′ O′ G′.

In a reinforced concrete beam, where the resistance of the concrete on the tensile side of the neutral axis is neglected, the diagram of the shearing stresses will take the form shown in Fig. 25, the shearing stress being a maximum at the neutral axis, and continues of the same amount until the tensile reinforcement is reached.

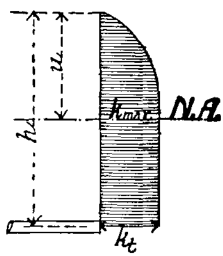


FIG. 25.

Considering a small vertical slice of a reinforced concrete beam, of a length x, acted upon by a total vertical shearing force K (Fig. 26), and if k_{max} is the maximum horizontal unit-shearing stress, k_t the unit bond stress between the concrete and the reinforcements, b is the width of the beam, O is the total perimeter of the reinforcing bars, C_1 and C_2 the total compressive stresses on the concrete on each side of the slice, F_1 and F_2 the total tensile stresses in the reinforcement on each side of the slice, C_2 and F_2 being the greater of the similar stresses, and $\left(h - \frac{u}{3}\right)$, as before, being the lever arm of the couple of resisting forces.

Now the total bond stress between the steel and concrete in the length of the slice x of the beam must be $F_2 - F_1$ and the unit bond stress

$$k_t = \frac{F_2 - F_1}{x\,O} \quad . \quad . \quad . \quad . \quad [1]$$

It will be seen from Fig. 26 that if equilibrium is to be maintained the moment K x must equal the moment $(F_2 - F_1)\left(h - \frac{u}{3}\right)$

or $$(F_2 - F_1) = \frac{K\,x}{\left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad . \quad [2]$$

Substituting [2] in [1], we get

$$*k_t = \frac{K}{O\left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad [3]$$

This gives the equation for the bond resistance.

Equation [3] may be written

$$k_t\,O = \frac{K}{\left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [4]$$

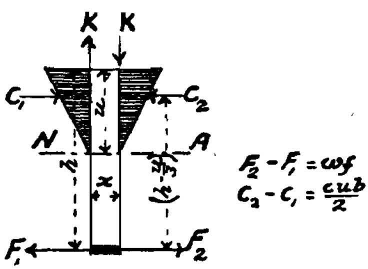


FIG. 26.

Now k_t O, or the total unit shearing or bond stress around the reinforcements, must be distributed over a horizontal section of unit length just above the plane of the reinforcements, and we have

$$O\,k_t = b\,k_{max}$$

which, substituted in equation [4], gives us

$$*k_{max} = \frac{K}{b\left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad [5]$$

This unit-shearing stress applies to all fibres below the neutral axis.

* See remarks, p. 124, as to the reduction of the value of K when inclined bars are used. For diagram giving values of $\left(h - \frac{u}{3}\right)$, *vide* "Manual of Reinforced Concrete, etc.," 1st edition, p. 147.

The above equations apply equally to doubly reinforced rectangular beams when the compressive reinforcement is placed at the centre of action of the compressive resistance of the concrete, or at a distance of $\frac{u}{3}$ from the compressive surface of the beam. When the compressive reinforcement is in any other position the value

$$\frac{m\, w_c\,(u - a)\,(h - a) + \frac{u^2 b}{2}\left(h - \frac{u}{3}\right)}{m\, w_c\,(u - a) + \frac{u^2 b}{2}}$$

must be substituted for $\left(h - \frac{u}{3}\right)$.

In the case of T beams the shearing resistance must be calculated as for a rectangular beam of the width of the rib.

Reinforcement against Diagonal Tension.

It is very usual to neglect the shearing resistance of the concrete when calculating the reinforcement against diagonal tension if it is found that the concrete is unable to resist it.

Inclined Reinforcements.

When an inclined reinforcement is employed to resist diagonal tension, if its area is ω_s, its direct resistance to tension is f, and the angle it makes with the horizontal is ϕ (Fig. 27).

The total direct resistance it offers is $\omega_s f$, and the resistance to vertical shearing will consequently be $\omega_s\, f\ sin\ \phi$ or $K = \omega_s f\, sin\ \phi$.

From this equation we get

$$\omega_s = \frac{K}{f\ sin\ \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad [6]$$

It follows from equation [6] that, when some of the longitudinal rods are bent up, or inclined reinforcements are used, if ω_s be their area, the value of K in equations [4] and [5] will become

$$K_1 = K - f\,\omega_s \sin\phi \quad . \quad . \quad . \quad . \quad . \quad [7]$$

If the vertical plane dropped from the upper extremity of an inclined shearing reinforcement falls outside the lower extremity of the next inclined reinforcement, the value of K in equation [6] must be the total shearing force on the length of the beam between the vertical plane and the lower extremity of the next reinforcement. This value of K will be the difference between the bending moments on the length of beam under consideration, as in the case of vertical reinforcements.

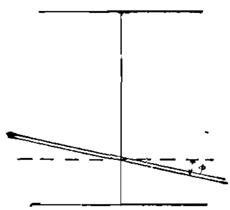


FIG. 27.

Vertical Reinforcements.

In this case each horizontal plane of the beam between the lower reinforcement and a plane close to the top is traversed by the vertical reinforcement. We must therefore find the total shearing force on the portion of the beam between neighbouring vertical reinforcements, and this, divided by the area of the vertical reinforcement and the lever arm of the longitudinal resisting forces, will give the maximum shearing stress on the reinforcements. If M_1 and M_2 are the respective bending moments, M_2 being the greater, then (referring to Fig. 26),

$$M_2 - M_1 = (F_2 - F_1) \times h - \frac{n}{3}, \text{ and } k_s\,\omega_s \text{ must equal } F_2 - F_1$$

$$\therefore M_2 - M_1 = k_s\,\omega_s \times \left(h - \frac{n}{3}\right)$$

$$\text{or } k_s = \frac{(M_2 - M_1)}{\omega_s \left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad [8]$$

This may be written

$$^{*}\omega_s = \frac{M_2 - M_1}{k_s \left(h - \frac{u}{3}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad [9]$$

The same remarks as before apply to the substitution for $\left(h - \frac{u}{3}\right)$ in the case of doubly reinforced beams (*vide* p. 123).

In equation [9], giving the area of the vertical shearing reinforcement, the value of k_s (the shearing resistance of the steel) may be taken as 12,800 lbs. per square inch.

When some of the longitudinal bottom reinforcements are bent up, the expression $(M_2 - M_1)$ must be replaced by $(M_2 - M_1) - f\, \omega_s\, x \sin \phi$.

To prove that $M_2 - M_1 = K$:—

We may consider a length of a beam to the left of the centre, between a section (A) at a distance x from the left support and a section (B) at a distance y from the left support, and using the following symbols—

R_L = reaction at the left support.

W = any load to the left of section (A).

W_1 = any load between the sections (A) and (B).

These loads being of any intensity

z = the distances of the loads (W) from section (A), and z_1 the distances of the loads (W_1) from section (B).

We have for the shearing force at section (A),

$$K_1 = R_L - \Sigma\, W,$$

* For method of employing this equation and table and diagram for use in obtaining the proper spacing, *vide* "Manual of Reinforced Concrete, etc."

and for the bending moment at section (A),

$$M_1 = R_E\, x - \Sigma\,(W\, z).$$

The mean value of the *decrease* of the shearing force on the length $(y - x)$ between the sections (A) and (B) will be,

$$\frac{\Sigma\,(W_1\, z_1)}{(y - x)}.$$

We have, therefore, for the mean value of the total shearing force on the length $(y - x)$

$$K_m = R \;- \Sigma\, W - \frac{\Sigma\,(W_1\, z_1)}{(y - x)}.$$

The total shearing force on the length $(y - x)$ will therefore be

$$K = R_L\,(y - x) - \Sigma\, W\,(y - x) - \Sigma\,(W_1\, z_1).$$

Now, the bending moment at section (B) will be

$$M_2 = R_L\, y - \Sigma\,(W\, z) - \Sigma\, W\,(y - x) - \Sigma\,(W_1\, z_1),$$

and the increase of the bending moment over the length $(y - x)$ will be

$$M_2 - M_1 = R_E\,(y - x) - \Sigma\, W\,(y - x) - \Sigma\,(W_1\, z_1) = K.$$

Resistance to Surface Cracking due to Temperature Stresses, Etc.

In a structure exposed to atmospheric influences the concrete near the surface, which is acted on by the temperature and moisture, tends to suffer deformation, but it is assisted in a great measure by the fibres behind it, which are protected from the direct action of atmospheric influences, and are therefore subject to less deformation.

The reinforcing metal is also protected, and will consequently be acted upon by a considerably less variation of temperature than that acting on the concrete.

It is for these reasons that temperature stresses in reinforced concrete structures can be assumed as being due to a considerably lesser range of temperature than those to which the exposed surface is subjected.

Concrete in damp situations will not tend to crack at low temperatures to the same extent as when in dry situations, since the effects of the dampness will counteract those of low temperature.

It is most important that parts of a structure which will be subjected to a considerable range of temperature should be constructed when the temperature is fairly low, or be thoroughly protected against the action of the sun's rays during hardening, as when these precautions are taken the effect of low temperatures will evidently be reduced.

For the reasons previously stated, and if reasonable precautions are taken during the hardening of the concrete, it is probably sufficient, in this country, to provide for a fall of temperature of, say, 30° Fahr. in the concrete and 20° Fahr. in the reinforcing steel.

The greatest range of temperature is about 70° Fahr. or from about 90° to 20°. Higher and lower temperatures occasionally occur, but they are exceptional and of short duration.

It is unnecessary to consider such a range, however, as, in the first place, the assistance offered to the concrete at the surface by the fibres behind has a considerable influence, and further, the concrete will be so protected by the moulds, etc., while hardening, that it will never reach a temperature approaching 90° if the moulds are kept damp. The evaporation will tend to lower the temperature acting on the concrete.

Assuming, therefore, as a maximum, a fall of temperature of 30° in the concrete and 20° in the steel, the deformation of concrete under temperature changes has been found by tests to be about 0·0000055 per degree Fahr.

The contraction for 30° Fahr., if the concrete were free, would therefore be $0{\cdot}0000055 \times 30 = 0{\cdot}000165$.

If the modulus of elasticity is taken at a high value of 3,000,000 lbs. per square inch, the tensile stress in the concrete, if prevented from contracting and not reinforced, would be $0{\cdot}000165 \times 3{,}000{,}000 = 495$ lbs. per square inch.

The reinforcing steel will be subjected to a fall of temperature of 20° Fahr. and has a deformation of 0·0000067 per degree Fahr. and a modulus of elasticity of 30,000,000 lbs. per square inch; therefore, as it may be considered as being held firmly at the ends, the induced tensile stress would be $0{\cdot}0000067 \times 20 \times 30{,}000{,}000 = 4{,}020$ lbs. per square inch.

If, therefore, the elastic limit of the steel is 32,000 lbs. per square inch, the resistance remaining to aid the concrete, if the steel is not stressed above its elastic limit, will be $32{,}000 - 4{,}020 = 27{,}980$ lbs. per square inch, and the ratio of area of steel required to the sectional area of concrete outside the bars will be $\frac{495}{27{,}980} = 0{\cdot}0177$, or 1·77 per cent.

The concrete will also offer some resistance to the induced tensile stresses, but this is neglected.

If high carbon steel were used with an elastic limit of 60,000, then the required ratio would be

$$\frac{495}{60{,}000 - 4{,}020} = \frac{495}{55980} = 0{\cdot}0088, \text{ or } 0{\cdot}88 \text{ per cent.}$$

Supposing, therefore, a parapet wall with a covering of $\frac{3}{4}$ inch outside the axis of the bars placed as reinforcement against temperature stresses, the following areas of steel will be necessary.

For mild steel an elastic limit of 32,000 per lbs. per square inch, $0·75 \times 12 \times 0·0177 = 0·160$ square inch per foot width, or $\frac{3}{8}$ inch round rods 8 inches apart.

And for high carbon steel with an elastic limit of 60,000 lbs. per square inch $0·75 \times 12 \times 0·0088 = 0·077$ square inch per foot width, or $\frac{1}{4}$ inch square bars 10 inches apart.

For resisting temperature stresses it is desirable to use high carbon steel, and the employment of bars giving a mechanical bond is also advisable.

The position and nature of a structure must be carefully considered with reference to temperature stresses, as the requisite reinforcement depends entirely on the conditions of each case, and no hard and fast rule can be laid down, although in most cases considerably less area of steel will be required than that found under the conditions assumed in the example which has been taken.

Pipes, Circular Reservoirs and Similar Structures.

When under Internal Pressure.

The direct tension on a unit length of the shell of a pipe or elevated circular reservoir or silo is given by the usual formulæ—

$$T = \tfrac{1}{2}p\delta, \quad . \quad . \quad . \quad [1]$$

where δ is the internal diameter, and p the unit pressure. If p is in pounds per square inch—

$$p = 0·43\ H_w,$$

H_w being the head of water in feet.

For a reservoir or water pipe, therefore—

$$T = 0·215\ H_w\delta, \quad . \quad . \quad . \quad [2]$$

T being in pounds on a circumferential strip one inch wide, H_w the head in feet, and δ the internal diameter in inches.

As the resistance of the concrete is neglected in tension we must have—

$$T = f\,\omega, \quad . \quad . \quad . \quad . \qquad [3]$$

f being the allowed unit stress in the reinforcement, and ω the sectional area of metal in the hooping reinforcement for the length of the structure taken.

If we take a circumferential strip with a width of (l) inches, all units being in inches and pounds, we get therefore from [1] and [3]—

$$\omega = \frac{lp\,\delta}{2f}, \quad . \quad . \quad . \quad . \qquad [4]$$

and from [2] and [3]—

$$\omega = \frac{0{\cdot}215\ H_w\,\delta\,l}{f}, \quad . \quad . \quad . \qquad [5]$$

from which we find the total sectional area of the hooping reinforcement for the length under consideration, which must be divided up into a suitable number of hoops or spirals.

For calculating the sectional area of the longitudinal bars, the portion of the shell between two adjacent hooping bars must be considered, this portion being treated as a slab built in at the ends, and of a span equal to the distance (L) between the hooping bars. For practical purposes the slab may be considered as flat between the two adjacent longitudinals.

As the shell and the longitudinals are continuous, we may consider the slab as securely fixed at the **ends**, and therefore the bending moments will be—

$$M_A = -\frac{1}{12}wL^2, \quad . \quad . \quad . \quad [6]$$

and

$$M_C = \frac{1}{24}wL^2. \quad . \quad . \quad . \quad [7]$$

At the hooping bars the concrete is in compression at the exterior of the shell, and the interior is in tension, while the reverse is the case at the centre of the span between the two hooping bars.

The longitudinal bars bear against the inside of the hoopings; it is therefore necessary to know the distances $t_{\prime}$ and $t_{\prime\prime}$ (Fig. 28), or the axes of the longitudinals from the surfaces of the supposed slab. This will give the position of the hooping reinforcement in the thickness of the shell. We also require the sectional area of the longitudinals.

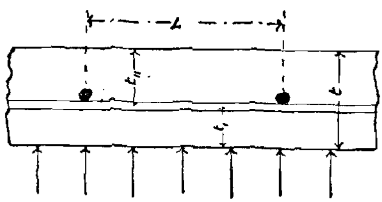


FIG. 28.

The thickness (t) of the shell is always decided upon from practical considerations, and in a great measure follows that which has been found good in practical examples. The thickness of shell for pipes seldom exceeds 3 to 3½ inches, and pipes up to 9 inches diameter are usually from 1½ to 2 inches thick. For reservoirs the thickness may be from 4½ to 6 inches, and sometimes more.

If we suppose the width under consideration (b) to be

4 inches since the shell is curved,* we already know the span L and the load w (being the pressure on the strip 4 inches wide). We can consequently assess value for M_A and M_C in equations [6] and [7]. Further, we have—

$$t_{\prime\prime} = (t - t_{\prime}). \quad . \quad . \quad . \qquad [8]$$

Now at the supports $M_A = \frac{1}{12}wL^2$, and since $b = 4$

$$\frac{M_A}{b} = \frac{wL^2}{3}. \quad . \quad . \quad . \qquad [9]$$

Similarly at the centre of the span

$$\frac{M_C}{b} = \frac{wL^2}{6}. \quad . \quad . \quad . \qquad [10]$$

If ω is the area of the longitudinals—

$$\psi_{\prime} = \frac{\omega}{bt_{\prime}} \text{ and } \psi_{\prime\prime} = \frac{\omega}{bt_{\prime\prime}}$$

Consequently $\psi_{\prime}\, bt_{\prime} = \psi_{\prime\prime}\, bt_{\prime\prime}$, or $\psi_{\prime}\, t_{\prime} = \psi_{\prime\prime}\, t_{\prime\prime}$, and from equation [8]—

$$\psi_{\prime}\, t_{\prime} = \psi_{\prime\prime}\,(t - t_{\prime}). \quad . \quad . \qquad [11]$$

We must now find values of $t_{\prime}$, $\psi_{\prime}$ and $\psi_{\prime\prime}$ from the diagram, Fig. 29, using the values of $\frac{M}{b}$ from equations [9] and [10], so as to satisfy equation [11].

It is well to try the economic percentage for $t_{\prime}$ and $\psi_{\prime}$ as a first trial.

Having found $\psi_{\prime}$ and $t_{\prime}$—

$$\omega = \psi_{\prime}\, bt_{\prime}. \quad . \quad . \quad . \qquad [12]$$

If we wish to place the bars more than 4 inches apart we can multiply ω by the proposed distance apart in

* Where the structure is of large diameter the value of b may be increased to 12 inches, the load w and the values for equations [9] and [10] being altered accordingly.

inches divided by 4. Having decided on the size of the longitudinals, the value of $t_{\prime}$ or $t_{\prime\prime}$ will give the position of the hooping reinforcement in thickness of the shell.

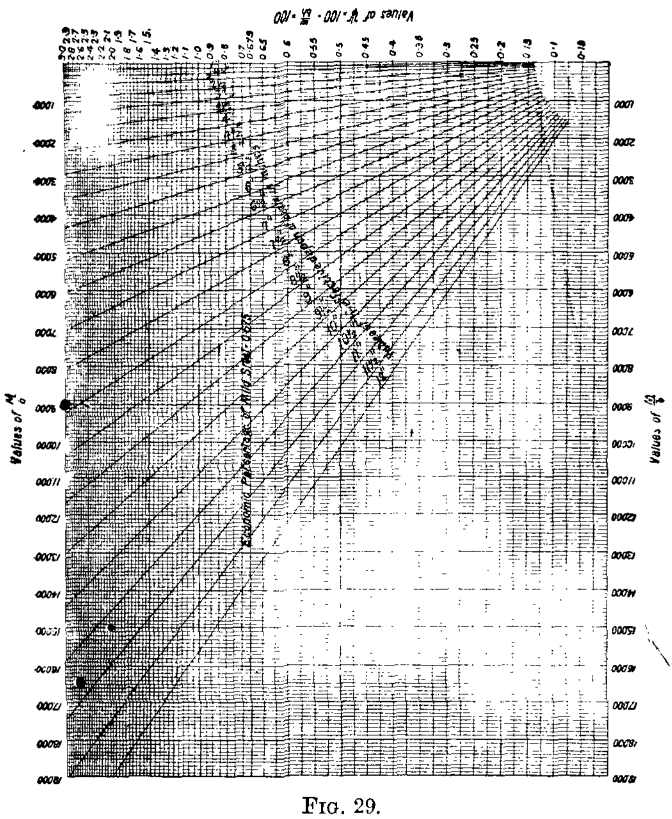


Fig. 29.

For thin pieces such as pipe shells, the hooping bars may be placed at the centre of the thickness and the areas of the longitudinals calculated under the worst conditions obtaining, which may be either at the support or at the centre of the span. In this case we have $\frac{M}{b}$ from equations [9] and [10], and also the values of $t_{\prime}$ and $t_{\prime\prime}$.

From Fig. 29 we obtain the values of $\psi_{\prime}$ and $\psi_{\prime\prime}$, from which the values of ω can be found under each condition. The greatest value so found must be used.

In the case of a pipe which has to bear transport, and handling while being deposited in the trench, it is well to somewhat increase the sizes of bars found by calculation, *for the same reason that we always increase the theoretical* thickness of a cast-iron pipe. This provision is, however, of less relative importance in the case of a reinforced concrete pipe, on account of the thickness of shell and nature of the reinforcement.

Many of the practical constructors only calculate for the hooping reinforcement and select a size for the longitudinals from practical experience without any calculation. If this course is adopted the hoopings should be placed at the centre of the thickness of the shell.

The hooping bars may be spirally wound or in the form of hoops, there being no *theoretical* advantage in the employment of either form, but there is a *practical* advantage in a spiral reinforcement, since there are fewer joints, and such a form is usually employed for small sections. Where the reinforcing skeleton is built up of rolled I, L, T or cross sections, and the concrete is poured into the moulds, the spiral form of hooping reinforcement allows the air to escape more easily and it is less likely to become imprisoned in the re-enterant angles; the concrete consequently surrounds the reinforcement more perfectly.

The longitudinals should be always secured to the circular reinforcement, as this helps to keep the latter in position. Where large pipes are used, and a double reinforcement is adopted, each set of longitudinals must of course be placed inside the hooping reinforcement to which it is attached.

The above method of treatment applies to pipes, circular reservoirs or tanks, silos, and similar structures, the only difference being that the pressure in pipes is uniform, while that in reservoirs, silos, etc., varies with the height. In the latter cases it is usual to consider the pressure as uniform over heights of 12 to 18 inches (on the Continent they usually take heights of 40 or 50 centimetres) and vary the sections as the depth decreases.

When under External Pressure.

In this case, as the piece is in compression, we may allow for the resistance of the concrete.

As before, we have the general formula—

$$P = \tfrac{1}{2} p \, \delta \quad . \quad . \quad . \qquad [13]$$

δ being the external diameter in this instance, P being the direct compressive stress on the shell. The method of treatment is the same as for the determination of the pieces under direct compression.

Taking for the value of

$$\psi = \frac{\omega}{lt} \quad . \quad . \quad . \quad . \qquad [14]$$

where ω is the sectional area of hooping reinforcement in a length (l) of the piece, and (t) is the thickness of the shell. We assume the limiting unit stress (c) on the concrete from which the area of the hooping reinforcement (ω) is deduced as shown in p. 98, either by assuming a thickness of shell or a value for (ψ). The unit stress on the concrete for ordinary proportions may be taken as 500 lbs. per square inch, but if a richer mixture is used a higher stress may be allowed; if, on the other hand, quick-setting cement is employed, this unit stress must be reduced.

The sectional area found for the hooping reinforcement must be divided up into a certain number of bars, which will fix their sectional area and spacing.

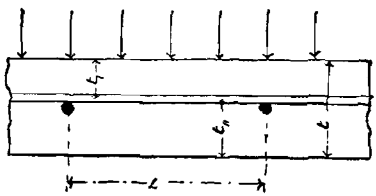


FIG. 30.

The calculation of the longitudinal bars is made in exactly the same manner as for structures under internal pressure, the longitudinals in this case being on the *outside* of the circular bars (Fig. 30). The tensile and compressive stresses are the reverse to those of a piece under internal pressure. The general remarks which have been made on the manner of treatment, etc., apply equally to pieces under external, as to those under internal, pressure.

SMALL SPAN ARCHES.

Arches with Uniformly Distributed Load, and Considered as Parabolic.

For small span arches, such as those used for floors, the arch may be considered as parabolic and the load as uniformly distributed. The curve of pressures is therefore parabolic.

If (w) is load per square unit, (L) the span, and (v) the rise, which will be practically the same as the versine of the neutral surface curve, (H) being the horizontal thrust we have—

$$Hy = \frac{wL}{2} + \frac{L}{4}$$

or

$$H = \frac{wL^2}{8v} \quad . \quad . \quad . \quad [1]$$

If we call the reaction at the springings (R_s)—

$$\text{Then } R_s = \sqrt{\left(\frac{wL}{2}\right)^2 + H^2}$$

or

$$R_s = \frac{wL^2}{8v}\sqrt{1 + \frac{16v^2}{L^2}} . \quad . \quad . \quad [2]$$

Both the horizontal thrust and the reaction at the springings will act at the neutral surface of the arch. The value of R_s being found, the equations for direct compression, pp. 97 and 98, must be used.

Herren Wayss and Fratag proceed as follows. The maximum compression being at the springing—

$$(d - \omega)\, c + \omega f = \frac{wL^2}{8v}\sqrt{1 + \frac{16v^2}{L^2}},$$

making $\omega = \psi\, bd$, and considering the width of the piece as unity, $\omega = \psi d$.

$$\text{Then } d = \frac{\frac{wL^2}{8v}\sqrt{1 + \frac{16v^2}{L^2}}}{c + \psi(f - c)}$$

taking the ratio of $\frac{L}{v}$ as 10, $d = 1{\cdot}35\, wL \times \frac{1}{c + \psi(f - c)}$ [3]

$$\text{and } \omega = \psi d. \quad . \quad . \quad . \quad [4]$$

Method for Arches Loaded over Half the Span and Considered Parabolic.

Another method employed for the calculations for arches is to consider the neutral line of the arch as parabolic, which is approximately true when the rise is small as compared with the span.

The dead load is supposed to be uniformly distributed. The live load is assumed to cover only half the span, as this loading causes the greatest bending moment. The curve of pressures for the dead load follows the curve of the arch, and that for the live load considered alone is supposed to pass through the neutral surface curve of the arch at the crown and springings. This is the same as assuming hinges at these places. In this case the thrust at the crown becomes—

$$H = \frac{L^2}{16v}(w + 2p), \quad . \quad . \quad [1]$$

and the maximum at the springings—

$$R = \frac{L^2}{16v}(w + 2p)\sqrt{1 + \frac{4v^2}{L^2}\left\{\frac{(3w + 4p)^2}{(w + 2p)^2}\right\}}, \quad [2]$$

w being the live and p the dead load.

The dead load produces no bending moment, as it is uniformly distributed and the curve of the arch assumed to be parabolic.

The maximum bending moment due to the live load only is produced at a section a quarter the length of the span from the springings, tending to cause a downward deflexion on the loaded side, and an upward deflexion on the unloaded side.

The ordinate of the parabolic pressure curve at the section $\frac{1}{4}$ L from the springing is $\frac{3}{4}\,v$. And the vertical component of the thrust at the springing on the unloaded side due to the live load only is (taking it as the reaction of a girder) $\frac{wL}{8}$. The horizontal thrust due to the live load only $= \frac{wL^2}{16v}$.

Taking moments, we get—

$$M_{max} = \frac{wL^2}{16v} \times \frac{3}{4} v - \frac{wL}{8} \times \frac{L}{4}$$

$$M_{max} = \frac{wL^2}{64} . \quad . \quad . \quad . \quad . \quad . \quad [3]$$

The thrust at $\frac{1}{4}$ L is $\frac{wL^2}{16v}(w + 2p)$ sec. ϕ, where ϕ is the angle of the neutral surface curve to the horizontal. The further treatment will be by the use of equations, from p. 186 forward.

These equations will apply to any arch hinged at the crown and springings if the weight of the arch and roadway can be considered as uniformly distributed, which is seldom the case in practice.

Large Span Arches and Other Pieces Subjected to Direct Stress and Bending Combined.

General Remarks.

In treating the question of pieces subjected to both direct and bending stresses the first essential is to know the position of the curve of pressures * through the piece

* The reasoning used in finding the position of the pressure curve, etc., follows closely that employed by Prof. William Cain in his "Elastic Arches" and "Steel Concrete Arches" published by W. Van Nostrand Company.

and the magnitude of the resulting pressures at different sections. When we have found the curve of pressures and its position on an *arch ring*, we may consider the force lines forming the pressure curve as acting at the vertical load lines. We have therefore at each of these sections a force R acting in the direction of the pressure curve at this point (Fig. 31). The effect of this force is not altered if we imagine two forces equal to R as acting at the neutral surface of the arch in opposite directions parallel to its line of action. This is the same as substituting for R a thrust at the neutral surface, and a couple with a lever arm equal to the perpendicular distance from the neutral surface to the pressure curve.

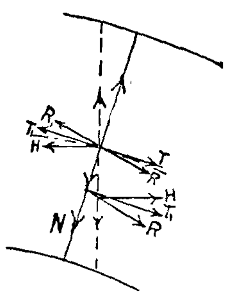


FIG. 31.

The thrust at the neutral surface may be resolved into components *tangential and normal to the neutral surface ;* the normal component only produces shearing, and is always small and is consequently negligible. The tangential component is the *direct* thrust, which we may call T.

The forces R and R of the couple producing the bending moment may also be resolved into *vertical and horizontal components.*

The vertical components act in opposite directions and therefore balance one another, and we have left a couple of horizontal forces with a lever arm of the vertical distance between the neutral surface and the pressure curve.

The horizontal force of the couple is the *horizontal* thrust, and is the same for all sections.

We have therefore the general equation for the bending moment M = H × the *vertical* distance from the neutral surface to the pressure curve, and varies at each section considered.

Now it will be seen that if at any section we were to resolve the forces R and R of the couple into components N normal and T_1 tangential to the neutral surface of the arch instead of the components acting in vertical and horizontal directions, the normal components would balance each other, and we should be left with a couple of forces T_1, and T_1, of the same magnitude as T but acting in the opposite direction to T, with a lever arm equal to the distance between the neutral surface and the pressure curve measured on the *radial* line of the arch, and this couple would produce a moment equal to T × the *radial* distance from the neutral surface to the pressure curve. Now, however, the forces R, R of the couple are resolved, the bending moment must remain the same, the components altering in magnitude, but the lever arm of the couple also varying inversely as the forces. Therefore T × the *radial* distance from the neutral surface to the pressure curve = H × the *vertical* distance from the neutral surface to the pressure curve = M.

We therefore get the relation $\frac{M}{T}$ = *radial* distance from the neutral surface to the pressure curve. In the case of columns or other pieces that are not curved there will be only one plane of reference in place of the radial and vertical planes of arches, also T will be the direct vertical thrust and there will be no expression similar to H. The relation $\frac{M}{T}$ will be equal to the *horizontal* distance from the neutral surface to the pressure curve.

Effect of the Bending Moments on an Arch Ring.

Consider a very small slice of an arch (Fig. 32) of a length Δs along the neutral surface, and having a central angle a. The direct thrust T cannot cause any change of curvature, but under the action of the bending moment M we may suppose the central angle changed to a_1, the curvature being increased if R acts below the neutral surface (as then the greatest compression is at the intrados) and decreased when R acts above the neutral surface—the angle $a_1 = a + \Delta a$.

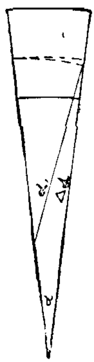


FIG. 32.

Therefore $\Delta a = (a_1 - a)$, . . [1]

and Δa is the change of inclination of the tangents to the curve due to the change of curvature, as is clearly seen by the exaggerated case (Fig. 33).

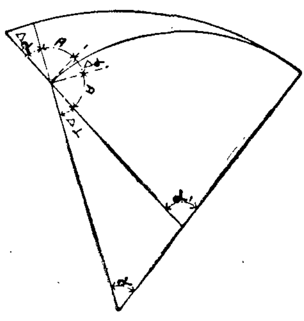


FIG. 33.

If we consider the bending moment as plus when it is left handed, then Δa is plus when M is plus or R acts above the neutral surface, and Δa is minus when M is minus, or R acts below the neutral surface.

If we call (v) the distance of any fibre of area (a) from the neutral surface, (v) being plus for fibres above and minus for fibres below the neutral surface, as the length of the arc is very small, it may be considered always as the arc of a circle and the axis of a fibre in the same plane as concentric with it.

Therefore the length of a fibre before flexure is $(\Delta s + va)$, and after flexure it becomes $(\Delta s + va_1)$.

The change of length is $v\,(a_1 - a)$, or from equation [1]—

$$\text{The change of length} = v\,\Delta a. \qquad [2]$$

If the unit stress of the concrete is (c), and of the metal is (f), since the unit stress on any fibre $= \frac{\text{elongation of fibre}}{\text{original length of fibre}} \times$ coefficient of elasticity, the stress on a fibre of concrete is—

$$c\text{a} = \frac{v\,\Delta a}{\Delta s + v\,a}\,\text{a}\,E_c, \qquad [3]$$

and on a fibre of the metal—

$$f\text{a} = \frac{v\Delta a}{\Delta s + v\,a}\,\text{a}\,E_f. \qquad [4]$$

The coefficient of elasticity of the concrete is here assumed to have a constant value.

The $(\Delta s + va)$ in the denominators may be replaced by Δs, without appreciable error.

The sum of all the stresses (due to flexure only) acting on the entire section is therefore—

$$\Sigma\,c\text{a} + \Sigma\,f\text{a} = \frac{E_c\,\Delta a}{\Delta s}\,\Sigma\,(v\text{a}) + \frac{E_f\,\Delta a}{\Delta s}\,\Sigma\,(v\text{a}). \qquad [5]$$

The moment of the stress on any fibre about the neutral surface must be (a c v) or (a f v) according as the fibre is of concrete or metal. Therefore the total bending moment = total resisting moment $= \Sigma\,v\,c\,\text{a} + \Sigma\,v\,f\,\text{a}$.

Then from equation [5]—

$$M = E_c \frac{\Delta a}{\Delta s} \Sigma (v^2 a) + E_f \frac{\Delta a}{\Delta s} \Sigma (v^2 a), \quad . \quad . \quad [6]$$

but $\Sigma (v^2 a)$ is the moment of inertia of the concrete or metal. We have therefore—

$$M = E_c \frac{\Delta a}{\Delta_s} I_c + E_f \frac{\Delta a}{\Delta_s} I_f,$$

$$\text{or } M = \frac{\Delta a}{\Delta_s} [E_c I_c + E_f I_f],$$

$$\text{Therefore } \Delta a = \frac{M \Delta s}{E_c [I_c + m I_f]} \quad . \quad . \quad . \quad . \quad . \quad [7]$$

We must now assume for the purpose of the graphical treatment *that for an appreciable length Δs, several feet for instance, Δa is given by equation* [7], *provided that* M *is taken as constant and equal to the value corresponding to that at the mid point of the length, or ½ Δs distant from either end,* I_c *and* I_f *being also taken there.* This assumption is very nearly true.

As the total change in the inclination of the end tangents for a length s is the sum of all the infinitesimal changes for the part of the arch under consideration, or

$$\Sigma \left(\frac{M \Delta s}{[E_c I_c + m I_f]} \right);$$

Δs being very small, the above assumption means that this expression is equal approximately to

$$\theta = \frac{M_c s}{E_c [I_c + m I_f]} \quad . \quad . \quad . \quad . \quad [8]$$

where $s = \Sigma \Delta s$ and M_c is the moment at the middle of s, I_c and I_f being also taken there.

If a, b, c (Fig. 34) represents the neutral surface line of an unstrained arch, and (s) a length of the neutral line whose centre is *b*. When the arch is loaded the neutral line changes shape, and the change of the inclination of the end tangents to the neutral arc *s* is given by equation [8], where M and E_c are constant, and M, I_c and I_f are taken at *b*.

Suppose the end *c* to be temporarily free, then the bending on *s* alone will cause a rotation of the arc *b c* about *b* equal to θ, so that the line *b c* will rotate through infinitesimal distance *c e*, taken as perpendicular to *b c*.

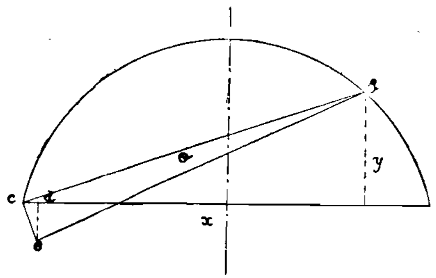


FIG. 34.

Taking *c* as the origin, and *c a* as the axis of *x* and the axis of *y* as vertical, and calling the co-ordinates of *b*, *x* and *y*, further drawing *e d* perpendicular to *c a*; then from similarity of triangles—

$$c\,d : c\,e :: y : b\,c, \text{ or } cd = \frac{ce}{bc}\,y \text{ and } \frac{ce}{bc} = \theta.$$

$$\text{Therefore } cd = y\theta. \quad . \quad . \quad . \qquad [9]$$

$$\text{similarly } de = x\theta. \quad . \quad . \quad . \qquad [10]$$

This assumes that if M, I_c, I_f, x and y are all taken at the mid point of the arc *s* as a sort of average, the horizontal and vertical deflections of *c*, due to *s*, are given with a sufficiently close approximation by the above equations.

The total horizontal and vertical displacements of *c* due to the bending of all the portions of the arch are then given by $\Sigma\,(y\,\theta)$ and $\Sigma\,(x\,\theta)$.

Further, if the tangent at (*a*) moves through at small angle β we have a vertical deflexion at (*c*), due to it, of

$\beta.\overline{ac}$ the horizontal displacement being *nil.* We have therefore from equation [8]

$$\Sigma\,(y\theta) = \Sigma\,\frac{\text{M}.s.y.}{\text{E}_c(\text{I}_c + m\text{I}_f)}$$

$$\text{and } \Sigma\,(x\theta) = \Sigma\,\frac{\text{M}.s.x.}{\text{E}_c(\text{I}_c + m\text{I}_f)} + \beta.\overline{ac}$$

The total *change* of inclination of the tangents at (*a*) and (*c*) is similarly—

$$\Sigma\theta = \Sigma\,\frac{\text{M}.s}{\text{E}_c(\text{I}_c + m\text{I}_f)}.$$

For an arch E_c is constant, and consequently, if we divide the neutral surface curve of the arch in such a manner as to make $\frac{s}{\text{I}_c + m\text{I}_f}$ constant—we can write

$$\Sigma(y\theta) = \Sigma\,\text{M}y. \qquad [11]$$

$$\Sigma(x\theta) = \Sigma\,\text{M}x + \beta.\overline{ac}. \qquad [12]$$

$$\text{and } \Sigma\theta = \Sigma\,\text{M}. \qquad [13]$$

We have then the following conditions—

When the arch is continuous having no hinges, we get—

$$\Sigma\,\text{M} = 0. \qquad [14]$$

$$\Sigma\,(\text{M}y) = 0. \qquad [15]$$

$$\Sigma\,(\text{M}x) = 0. \qquad [16]$$

When the arch is hinged at the springings, the span is invariable. Therefore—

$$\Sigma\,\text{M}y = 0. \qquad [17]$$

The vertical deflexion of *c* with respect to *a* is zero, but β will have a value; therefore Σ (M x) cannot be zero.

Pressure Curve.

General Remarks.

The general principle employed for finding the true pressure curve on any arch due to the loading and methods of fixing, whether hinged or otherwise, is stated as follows by Professor Cain, in his "Elastic Arches."*

"If in any arch the equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from consideration of the conditions imposed by the supports, etc.; and if, furthermore, the neutral surface curve of the arch itself be regarded as another equilibrium polygon due to some systems of loading not given, and its closing line be also found from the same considerations respecting supports, etc.; then when these two polygons are placed so that their closing lines coincide and their areas partially cover each other, the ordinates intercepted between the two polygons are proportional to the real bending moments acting in the arch."

We have also, as a principle of the equilibrium polygon, that if the ordinates have to be altered in a given ratio, the pole distance is altered in the inverse ratio. This simply means that if the slope of the lines in the diagram of forces is to be altered, the vertical forces remaining the same, it is necessary to increase the pole distance for a flatter slope, and decrease it for a steeper slope.

If the springings are at different levels we have, instead of a horizontal thrust, a thrust parallel to the line joining

* The methods employed by Professor Cain are followed in the paragraphs dealing with the location of the pressure curve, by his permission. These methods the reader is given in "Elastic Arches," "Concrete Steel Arches and Theory of Solid and Braced Elastic Arches," published by Van Nostrand Co., 23, Murray Street, New York, at 50 cents each.

the springings. In the discussion to follow, it is always assumed that the springings are at the same level.

Practical examples have been given in the following methods for finding the true position of the pressure curves in arches, as it was considered that this would be preferable to a general treatment. Different forms of arched bridge and loading are given in each case in order that the whole process may be clearly shown. The bridge with arched spandrils assumed for the two-hinged arch can be adapted to the other types, the only difference being that the load lines or P_s are varied in position. For the three-hinged arch, a bridge with a rise of $\frac{1}{5}$ the span has been assumed instead of the flatter arch selected for the other types.

In all the examples the exterior load has been assumed as covering half the span, but when designing a bridge several positions of the exterior load should be tried. It is usual to assume two cases, one when the load covers the whole span, and the other when it covers half the span, but Mr. Cain, in the discussion on a paper recently read before the American Society of Civil Engineers by Mr. B. R. Leffler,* advises the trial of two further positions, one extending from the springing $\frac{3}{10}$ and the other $\frac{6}{10}$ of the span.

Continuous Arch having no Hinges (Fig. 39).

The description and loading of the assumed bridge is shown on the figure, the weight of the concrete used in the spandrils is assumed to be the same as that of the

* "Transactions of American Society of Engineers," vol. lv.

arch for the sake of simplicity, although it is usually less. The dimensions of the arch ring were found by the use of a formula devised by F. F. Weld,* based upon the study of all available data upon the subject of his own experience in designing arches for a great variety of conditions of span and load. This formula may be stated as follows—

$$d = \sqrt{L} + 0{\cdot}1L + 0{\cdot}005w + 0{\cdot}0025p.$$

Where d is the depth of the arch at the crown in inches.

L the clear span in feet.

w the exterior load in pounds per square foot uniformly distributed.

p the weight of the dead load above the crown of the arch per square foot in pounds.

It may be of interest to mention that an exterior load of 200 lbs. per square foot will allow for a 15 ton steam roller.

The radial depth of the arch ring at the quarter points should be $1\frac{1}{3}$ that at the crown.

The extrados curve may be struck passing through the points obtained as above from a centre on the line from the crown passing through the centre from which the intrados was struck, as has been done in the example given, or the radial depth of the arch at the springings may be made double that at the crown, the outer portions of the extrados being drawn in tangent to the curve obtained, as described above.

An elliptical form of arch is frequently adopted when there are no hinges and the rise is about $\frac{1}{5}$ the span. In this

* *Engineering Record*, November 4, 1905.

case the intrados is described in the usual way with five centres or as a true ellipse; the depth at the crown and quarter points is found as described for segmental arches.

A neat method for the graphic construction of a five-centred curve is given by Mr. A. Swartz in a letter to the *Engineering Record.*

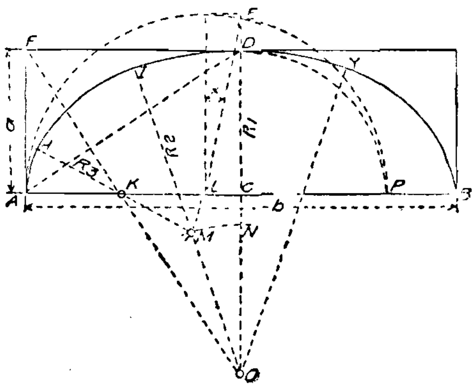


FIG. 35.

Since we know the rise and span; let A B, Fig. 35, represent the span and C D the rise. Draw D F parallel to A B and produce D C. Join A D. Draw F O perpendicular to A D. Make C P = C D, and describe a semi-circle on A P, cutting C D produced at E. Make C N = E D and describe the arc M N from the centre O. Make A L = C E, and describe the arc L M from the centre K, cutting M N at M. K, M and O are the centres, and A K, M H and O I the radii.

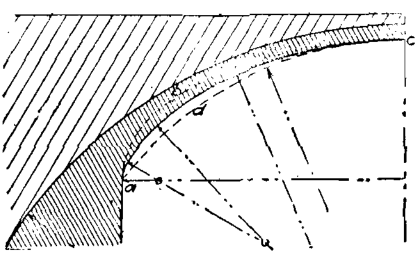


FIG. 36.

Mr. Daniel B. Luten, President of the National Bridge Company, in an article published in the *Engineering*

Record of April 14, 1906, discussing the best form of arch ring for reinforced concrete bridges, advises a curve for the intrados of a mean between the ellipse and the segment as shown in Fig 36. The ellipse is readily determined by drawing two concentric circles on the major and minor axes of the ellipse as diameters, and from the points where any common radius cuts the two circles, projecting lines parallel to the respective axes. The points of intersection of these lines will be points on the ellipse.

Points on the curve of the intrados are found by bisecting the vertical distances between the ellipse and the segment and determining by trial arcs of circles to approximate the true curve.

The curve of the extrados is a segment struck from a centre on the centre line of the arch, with a radius equal to the radius of the intrados, at the crown, plus 2½ times the thickness of the arch at the crown.

Mr. Luten states that an arch designed as above will be in almost exact equilibrium under earth loading, when the depth of fill at the crown does not exceed three times the crown thickness. Where the fill exceeds this amount, it will nearly always occur in high embankments where the semi-circle is feasible and is the most efficient curve.

With a concentrated load at the crown the above form of arch will be found to require reinforcement near the intrados at the crown, and near the extrados at the haunches. By using one series of reinforcements for both these regions and alternating the points of their crossing the arch ring, one system of rods can be made to reinforce the arch against all the stresses.

By distributing the points of bending so that the rods will cross the middle and thirds of the half arch, this reinforcement will provide for all possible concentrations of loading.

One arch of a double span bridge constructed at Franklin, Indiana, with an arch ring and reinforcement as described is shown (Fig. 37), and Mr. Luten states that numerous arches up to 100 feet span have been built in this manner with invariable success.

Having drawn the arch ring, the neutral surface curve must be drawn in passing through the mid-points of the

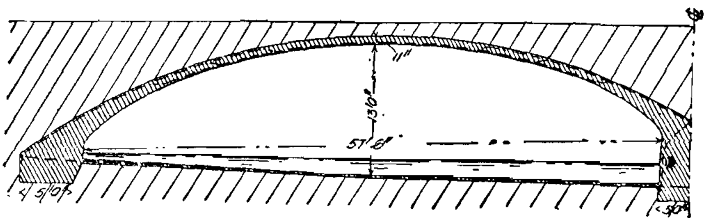


FIG. 37.

thickness of the arch ring. This line must now be divided into such parts that if s is their respective length in feet, and d the respective depths at the centres of the lengths s in feet, I_c the moment of inertia of the concrete in foot units, I_f the moment of inertia of the reinforcement in foot units, and m the ratio of the moduli of elasticity or $\frac{E_f}{E_c} = 15$, then $\frac{s}{I_c + mI_f}$ must be constant; or, considering a longitudinal strip of the arch 12 inches wide, as $I_c = \frac{bd^3}{12}$, we must divide the neutral surface curve into such lengths s that $\frac{s}{d^3 + 15I_f}$ is constant.

We can scale the depths of the arch ring d, but in order to find I_f a percentage of metal reinforcement must be assumed.

It will be usually sufficiently accurate if this percentage is taken as 0·75 *per cent. of the area of the* 12 *inch width of the arch ring at the crown.*

This will be distributed half near the intrados and half near the extrados.

This area of metal may have to be increased or reduced when finally calculated, but the adoption of this percentage will be quite sufficiently accurate for obtaining the divisions of the neutral surface curve.

In the case considered in Fig. 39, p. 170, the depth of arch at the centre found by equation [1] is 1·25 feet. Therefore the sectional area of reinforcement in a section of the arch 1 foot wide, will be $1·25 \times 0·0075 = 0·0094$ square feet. The reinforcements will be placed with their axes say 2 inches from the intrados and extrados respectively. Consequently as the neutral surface lies half-way between the intrados and extrados the value of the moment of inertia of the reinforcements will be $I_f = 0·0094 \times \left(\frac{d}{2} - 0·17\right)^2$. The depth d varies at each of the sections of arch ring considered.

Now to effect the proper division of the neutral surface line. The depth of the arch ring at the springing and the centre and also at several points along the neutral surface line from the springing to the centre (say 5 points) are scaled off as accurately as possible ; it is advisable to plot the scaled depths as ordinates to an enlarged scale from a horizontal line representing the half length of the neutral surface curve straightened out, and draw an even curve passing through or close to the points plotted and so adjust the depths, using those scaled from the curve thus obtained for the purposes of calculation.

Table IX. gives these depths together with the values of d^3, $m\mathrm{I}_f$, and $\frac{1}{d^3 + m\mathrm{I}_f}$.

TABLE IX.

(1) Distance along neutral surface line from springing. Feet.	(2) Depth of arch ring d. Feet.	(3) d^3.	(4) $m\mathrm{I}_f = 15 \times 0{\cdot}0094 \times \left(\frac{d}{2} - 0{\cdot}17\right)^2$.	(5) $\frac{1}{d^3 \times m\mathrm{I}_f}$.
0	2·85	23·149	0·223	0·043
4	2·45	14·706	0·158	0·067
8	2·12	9·528	0·112	0·104
12	1·85	6·332	0·081	0·156
16	1·625	4·291	0·059	0·230
20	1·45	3·049	0·044	0·323
24	1·32	2·300	0·034	0·428
28	1·26	2·000	0·030	0·493
31·12 at centre	1·25	1·953	0·030	0·504

Now draw a horizontal line AB (Fig. 38) equal to the length of half the neutral surface curve (31·12 feet, in the case under consideration) and divide this at the points at which the above depths have been taken, *i.e.*, at 0, 4, 8,...28 and 31·12 feet from the springing end. Erect perpendiculars at each of the points of division and set off on each perpendicular, to any convenient scale, the values $\frac{1}{d^3 + m\mathrm{I}_f}$.

These are given in the 5th column of Table IX. Join the points so laid off, forming a curve and ink in this curve together with the perpendiculars AC and BD at the

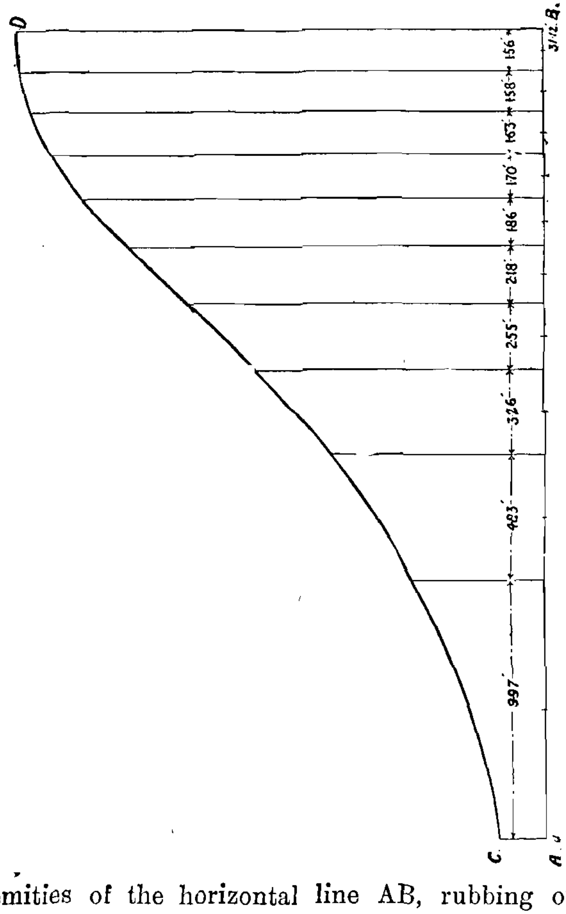


Fig. 38.

extremities of the horizontal line AB, rubbing out the intermediate perpendiculars.

Now if the area ABDC is divided into equal parts by perpendicular lines, these perpendiculars will cut the horizontal line at points which will divide it in such a way that if the half neutral surface line is divided in a like manner, $\frac{s}{I_c + mI_f}$ will be constant.

This can readily be done by a few trials as the upper curved line will be approximately straight between the several divisions, excepting that nearest the springing.

The figure being divided into the desired number (say 10) of vertical strips, the area of each of which is, in this case, equal to one-tenth the whole area of ABDC. This division is shown in Fig. 38.

In the case of the example (Fig. 39, p. 170) Table X. gives the lengths of the divisions found as described above, their summations and the distances from the springing to the centres of the divisions giving the points a, a_2, a_3, etc.

TABLE X.

Length of Division of Neutral Surface. Curve from Springing in feet.	Summation in feet.	Distance from Springing to Centres of Divisions in feet.
9·97	9·97	4·98
4·83	14·80	12·39
3·26	18·06	16·43
2·55	20·61	19·34
2·18	22·79	21·70
1·86	24·65	23·72
1·70	26·35	25·50
1·63	27·98	27·16
1·58	29·56	28·77
1·56	31·12	30·34

In the example the exterior load is assumed as covering half the arch, as this gives approximately the maximum deviation for the pressure curve from the neutral surface curve.

It is usual to assume three cases: the load covering half the span, the load covering the whole span, and with no load on the bridge as the relative thrusts at the springings will affect the design of the piers and abutments, and the load covering half the span and covering the whole span give different sections for maximum bending, and also the thrusts on the sections vary in intensity in these two cases.

It must also be remembered that change of temperature has considerable influence, and consequently the calculation of the bending moments and direct stresses due to this cause must never be neglected.

Having set off the a^s along the neutral surface curve perpendiculars are dropped from these points and the loads for a strip of the bridge and loading one foot in width are found for the portions of the arch, superstructure and exterior load between them. At the crown two loads are found between a_{10} and a_{11} and the centre line. These loads act along the lines P passing through the centres of gravity of the several trapezoids enclosed between the verticals through the a^s, the intrados of the arch, and the line of loading, equivalent load lines being drawn reducing all the several loads to the same value.

It is only necessary to find the centre of gravity for the lines of action for the P^s where the inclination of the intrados is considerable, in the present case for P_1, P_2, P_{19} and P_{20}, the remaining P^s may be drawn acting through the centre of the distance between the a^s.

The loads for the outer portions between a_1 and a_{20} and the springings, and acting through the centres of gravity of these outer portions, are not used for obtaining the equilibrium polygon, but are compounded with the thrusts through a_1 and a_{20} to obtain the thrusts through springings R_B and R_A.

TABLE XI.

	Load.	Summation.	For obtaining ordinate for plotting equilibrium curve.	
			Additions from right and left from P_{12} and P_{11}.	Distances between P_s.
P_1	7,087	7,087	18,033	5·78
P_2	2,925	10,012	10,946	
P_3	1,793	11,805	8,021	3·35
P_4	1,359	13,164	6,228	2·55
P_5	1,051	14,215	4,869	2·09
P_6	907	15,122	3,818	1·82
P_7	816	15,938	2,911	1·72
P_8	751	16,689	2,095	1·62
P_9	740	17,429	1,344	1·55
P_{10}	382	17,811	604	1·13
P_{11}	222	18,033	222	0·78
P_{12}	430	18,463	430	
P_{13}	441	18,904	871	1·55
P_{14}	486	19,390	1,357	1·62
P_{15}	551	19,941	1,908	1·72
P_{16}	671	20,612	2,579	1·82
P_{17}	889	21,501	3,468	2·09
P_{18}	1,243	22,744	4,711	2·50
P_{19}	2,165	24,909	6,876	3·40
P_{20}	5,587	30,496	12,463	5·90

Table XI. gives the values of P_1, P_2, etc., together with these summations.

The forces are now laid off to a scale of loads on the vertical load line, and a trial horizontal thrust of 20,000 lbs. has been assumed acting between P_{11} and P_{12}, giving a trial pole O_1*. Now join the points on the vertical load line at the terminations of the several P^s to O_1.

* The trial horizontal thrust should preferably be assumed as acting between P_{10} and P_{11} for the reason given on p. 171.

The rays from the points between the several P^s give the amount and direction of the several thrusts between those P^s in the equilibrium polygon, *i.e.*, the ray from the point on the load line between P_1 and 6,682 gives the thrust acting between the outer vertical load of 6,682 and P_1, in amount and direction; similarly that from the point between P_1 and P_2 gives the thrust between P_1 and P_2, calling these rays P_1, P_2, P_3, P_4, etc. We commence drawing the equilibrium polygon by drawing a horizontal line between the load lines P_{11} and P_{12} (in the present case this is taken on O_1 H_1 produced), since the horizontal thrust has been assumed as acting between these.

The polygon is completed by drawing lines between the various load lines parallel to the respective rays, *i.e.*, draw lines from the ends of the horizontal between P_{11} and P_{12}, from P_{11} parallel to the ray P_{11} P_{10}, and from P_{12} parallel to the ray P_{12} P_{13} from where these lines cut P_{12} and P_{13} lines parallel to the rays P_{10} P_9 and P_{13} P_{14}, till they cut the load lines P_9 and P_{14}, proceeding in this manner until the rays $P_1 6682$ and $P_{20} 5602$ cut the vertical lines dropped from a_1 and a_{20} at v_1 and v_{20}.

The equilibrium polygon can be checked by laying off vertical ordinates from the horizontal through P_{11} P_{12} on the several load lines. The ordinates being found as follows :—

From similar triangles P_{12} : H_1 O_1 : : the ordinate from the horizontal through P_{11} P_{12} at P_{13}: the horizontal distance from P_{12} to P_{13}; or the ordinate on P_{13}

$$= \frac{P_{12} \times \text{horizontal distance between } P_{12} \text{ and } P_{13},}{H_1 O_1}$$

$$\text{consequently the ordinate} = \frac{430 \times 1{\cdot}55}{20{,}000} = 0{\cdot}03.$$

Also $(P_{12} + P_{13}) : H_1 O_1 :: $ the ordinate from the horizontal through P_{13} : the horizontal distance from P_{13} to P_{14}.

∴ The ordinate at P_{14} from the horizontal through $P_{11} P_{12}$ = ordinate on P_{13}

$$+ \frac{(P_{12} + P_{13}), \times \text{horizontal distance between } P_{13} \text{ and } P_{14},}{20{,}000}$$

$$\text{or ordinate} = 0{\cdot}03 + \frac{871 \times 1{\cdot}55}{20{,}000} = 0{\cdot}03 + 0{\cdot}067 = 0{\cdot}10.$$

Similarly $(P_{12} + P_{13} + P_{14}) : H_1 O_1 :: $ the ordinate from the horizontal through P_{14} : the horizontal distance from P_{14} to P_{15}.

∴ The ordinate at P_{15} from the horizontal through $P_{11} P_{12}$ = ordinate at P_{14}

$$+ \frac{(P_{12} + P_{13} + P_{14}) \times \text{horizontal distance between } P_{14} \text{ and } P_{15},}{20{,}000}$$

$$\text{or ordinate} = 0{\cdot}10 + \frac{1{,}357 \times 1{\cdot}62}{20{,}000} = 0{\cdot}10 + 0{\cdot}109 = 0{\cdot}21,$$

and so on.

These ordinates must be laid off to the scale of distances.

Having plotted the equilibrium polygon join $v_1 v_{20}$ and produce the verticals through a_1 a_2, etc., to cut the equilibrium polygon b_1, b_2, etc., and the line joining $v_1 v_{20}$ at v_1, v_2, v_3, etc.

The ordinates b_1v_1, b_2v_2, b_3v_3, etc., are those which must be used for the purpose of finding the true line of resistance. We must first find the true closing line for the equilibrium polygon to satisfy the conditions $\Sigma M = 0$, and $\Sigma Mx = 0$ (p. 146).

The bending moments are proportional to the ordinates of the equilibrium polygon from the closing line since the horizontal thrust is constant. Consequently if FF_1 be the closing line, $\Sigma M = 0$ will be satisfied if the algebraical sum of the ordinates from FF_1 to b_1, b_2, etc., or $\Sigma\ (fb)$ is zero, those measured downwards from the b^s being positive and those measured upwards negative. If the sum of the ordinates between FF_1 and v_1, v_2, v_3, etc., are added to the above equality, we get the condition that the sum of the ordinates between FF_1 and v_1, v_2, v_3, etc., shall equal the sum of the ordinates v_1b_1, v_2b_2, v_3b_3, etc., or that

$$\Sigma(vf) \text{ must } = \Sigma(vb). \quad . \quad . \quad . \quad [A]$$

Also since $\Sigma(vb) - \Sigma(vf) = \Sigma(fb)$, the second condition $\Sigma(Mx.) = 0$ may be written $\Sigma(fb.x) = \Sigma(vb.x) - \Sigma(vf.x) = 0$, or $\Sigma(vb.x) = \Sigma(vf.x)$.

This indicates that if the ordinates of the types vb and vf are regarded as forces, the sums of their moments about an abutment are equal, or that the resultant of the vb^s coincides with that of the vf^s, since $\Sigma(vb) = \Sigma(vf)$ from equation [A].

Consequently if we make the resultants of the vb^s and vf^s treated as forces coincide, we satisfy the second condition that $\Sigma(M.x) = 0$, and the first condition that $\Sigma M = 0$ is satisfied if we obtain the equality $\Sigma(vf) = \Sigma(vb)$ of equation [A].

To find the true position for the closing line FF_1 to satisfy these conditions, we first find the resultant of the ordinates of the type vb treated as forces in position and magnitude. The length of the sum of the ordinates vb is most easily found by marking off the several lengths in succession on the edge of a piece of paper and scaling the total length thus found. Any scale can be used (generally the scale of lengths) provided the same scale is used for measuring the ordinates throughout the whole subsequent process.

Calling the magnitude of the resultant of the vb^s R we find R = 125·95.

To find the position of this resultant we take moments of all the vb^s about D on the vertical through the crown. This may be conveniently done by measuring the horizontal distances from the vertical through D to each of the vb^s in hundredths of a foot, calling these z_2, z_3, etc.

And dividing $[(v_{19}b_{19} - v_2b_2)z_2 + (v_{18}b_{18} - v_3b_3)z_3 + v_{17}b_{17} - v_4b_4)z_4 + \ldots + (v_{11}b_{11} - v_{10}b_{10})z_{10}] = 50 \cdot 04 \times R = 125 \cdot 95$, as shown in Table XII., we find that R acts 0·40 feet to the right of D.

TABLE XII.

1 Length of ordinates.	2 Length of ordinates.	3 Sum.	4 Difference.	5 Horizontal distances from C D to $b.vs$ in succession from C D to v_2b_2.	6 Products of col. 4 and col. 5.
$v_2b_2 = 4·73$	$v_{19}b_{19} = 4·06$	8·79	0·67	$z_2 = 18·22$	12·207
$v_3b_3 = 6·28$	$v_{18}b_{18} = 5·48$	11·76	0·80	$z_3 = 14·33$	11·464
$v_4b_4 = 7·06$	$v_{17}b_{17} = 6·24$	13·30	0·82	$z_4 = 11·53$	9·455
$v_5b_5 = 7·53$	$v_{16}b_{16} = 6·79$	14·32	0·74	$z_5 = 9·28$	6·867
$v_6b_6 = 7·81$	$v_{15}b_{15} = 7·16$	14·97	0·65	$z_6 = 7·31$	4·751
$v_7b_7 = 7·97$	$v_{14}b_{14} = 7·41$	15·38	0·56	$z_7 = 5·55$	3·108
$v_8b_8 = 8·01$	$v_{13}b_{13} = 7·61$	15·62	0·40	$z_8 = 3·94$	1·576
$v_9b_9 = 8·05$	$v_{12}b_{12} = 7·82$	15·87	0·23	$z_9 = 2·33$	0·536
$v_{10}b_{10} = 8·02$	$v_{11}b_{11} = 7·92$	15·94	0·10	$z_{10} = 0·78$	0·078
Summation		125·95 = R			50·042

$$\frac{50·04}{125·95} = 0·40 \text{ or R acts 0·40 feet to the right of C D.}$$

$$v_{20}E = v_1E_1 = \frac{R}{n} = \frac{126}{20} = 6·30 \text{ to scale of distance.}$$

Now we have to find a closing line from which the sum of the ordinates to $v_1v_{20} = 126$, and where the resultant of these ordinates considered as forces acts at 0·40 feet to the right of D.

Assuming a closing line EE_1 making $v_{20}E$ and v_1E_1 $= \frac{R,}{n}$ where n is the number of ordinates (20 in the present case) $\therefore v_{20}E = v_1E_1 = \frac{126}{20} = 6·30$.*

Next join Ev_1, dividing the ordinates from EE_1 to $v_{20}v_1$ into two sets.

Now since $v_{20}E = v_1E_1$, the resultant of the ordinates within the triangle $v_{20}Ev_1$ treated as forces will equal that of the ordinates within the triangle v_1E_1E in magnitude, each being equal to $\frac{R}{2} = \frac{126}{2} = 63·0$ since $v_{20}E = v_1E_1$ $= \frac{R,}{n}$ and these resultants act at the same distance from D, the one to the right and the other to the left, calling the resultant to the left of D Trial T, and that to the right of D Trial T_1.

Next find the position of Trial T by taking moments about D in the same manner as that used in finding the position for R.

The difference between the ordinates at the same distance from CD may be conveniently marked off by joining the point where Ev_1 cuts CD to v_{20}, and measuring the portion of the ordinate above this line.

The z^s will be the same as those used for finding the position of R. The working is shown in Table XIII.,

* Any other position for EE_1 would do, but the assumed position makes the subsequent working easier. It is recommended by Professor Cain in his discussion on Mr. Leffler's paper mentioned previously.

M 2

Trial T being found to act 5·94 feet to the left of D, and Trial T_1 acting 5·94 feet to the right of D.

TABLE XIII.

Horizontal distance from CD to vEs in succession from CD to v_{20}E.	Difference of ordinates for Trial T.	Products.
$z_{11} = 0·78$	0·21	0·164
$z_{12} = 2·33$	0·59	1·375
$z_{13} = 3·94$	0·98	3·862
$z_{14} = 5·55$	1·38	7·659
$z_{15} = 7·31$	1·80	13·159
$z_{16} = 9·28$	2·28	21·158
$z_{17} = 11·53$	2·81	33·523
$z_{18} = 14·33$	3·50	50·153
$z_{19} = 18·22$	4·48	81·625
$z_{20} = 25·68$	6·30	161·784
Summation		374·462

$$\text{Trial T} = \text{Trial T}_1 = 63·00 \text{ and } \frac{374·46}{\text{Trial T}} = \frac{374·46}{63·00} = 5·94$$

Trial T acts 5·94 feet to the left of CD,
and Trial T_1 acts 5·94 feet to the right of CD.

$$* \ v_{20}F = \frac{2l_1}{l + l_1} \times \frac{R}{n} = \frac{2 \times 5·54}{11·88} \times \frac{125·95}{20} = 5·87$$

$$* \ v_1F_1 = \frac{2l}{l + l_1} \times \frac{R}{n} = \frac{2 \times 6·34}{11·88} \times \frac{125·95}{20} = 6·72.$$

Now the position of Trial T is not changed if E is moved to F, since the ordinates are all altered in the same ratio, and similarly the position of Trial T_1 is not altered if E_1 is moved to F_1.

Now since R is the resultant in magnitude and position of Trial T and Trial T_1 we have, by taking moments in turn about t_1 and t on the lines of action of T_1 and T.

Since R = 2 Trial T = 2 Trial T_1.

* As shown on p. 165.

True T $\times$ 11·88 $=$ R(5·94 $-$ 0·40) $=$ 2 Trial T $\times$ 5·54, and similarly True $T_1 \times$ 11·88 $=$ 2 Trial $T_1 \times$ 6·34,

$$\text{or} \quad \frac{\text{True T}}{\text{Trial T}} = \frac{2 \times 5{\cdot}54}{11{\cdot}88}, \text{ and } \frac{\text{True } T_1}{\text{Trial } T_1} = \frac{2 \times 6{\cdot}34}{11{\cdot}88}.$$

$$\text{Consequently } v_{20}\,F = \frac{\text{True T}}{\text{Trial T}} v_{20}\,E = \frac{2 \times 5{\cdot}54}{11{\cdot}88} \times \frac{R}{n}$$

$$= \frac{2 \times 5{\cdot}54}{11{\cdot}88} \times \frac{125{\cdot}95}{20} = 5{\cdot}87.$$

$$\text{and } v_1F_1 = \frac{\text{True } T_1}{\text{Trial } T_1} v_1\,E_1 = \frac{2 \times 6{\cdot}34}{11{\cdot}88} \times \frac{R}{n} = \frac{2 \times 6{\cdot}34}{11{\cdot}88} \times \frac{125{\cdot}95}{20} = 6{\cdot}72.$$

Set off $v_{20}F$ $=$ 5·87 and v_1F_1 $=$ 6·72, and join FF_1, giving the true closing line to satisfy the necessary conditions.

The next step is to find the closing line to the neutral surface curve of the arch regarded as an equilibrium polygon, due to some system of loading not given, so as to satisfy the conditions $\Sigma M = 0$ and $\Sigma(M.x) = 0$.

Suppose KK_1 is this closing line, then since the curve is symmetrical the resultant of the ordinates from the line AB (joining the ends of the neutral surface line) to the neutral surface curve (the y's on the figure), treated as forces passes through the crown.

Therefore to satisfy the condition $\Sigma(M.x) = 0$ the resultant of the ordinates from AB to KK_1 must also pass through the crown.

This can only occur when KK_1 is parallel to AB. Therefore in this case KK_1 must be a horizontal line. To satisfy the condition $\Sigma M = 0$ the algebraical sum of the ordinates from the line KK_1 to the neutral surface curve must be zero, those measured upwards from KK being positive, and those measured downwards being negative.

It is therefore necessary to place the line KK_1 at a distance from AB, equal to the mean length of the ordinates of the type y.

Since the neutral surface curve is symmetrical about the centre, if we take the sum of the y^s on one side of the centre and divide by half the number of ordinates or the number of ordinates on one side of the centre, we get this mean length—

The sum of the ordinates for half the arch = 44·23, which divided by 10 = 4·42 = e, or the distance from AB to KK_1. We therefore draw in KK_1 parallel to AB (or horizontal), and 4·42 feet to the scale of distance from AB.

We have now found the true closing lines for both the equilibrium curve and the neutral surface curve of the arch.

Now if we imagine the closing line FF_1 to be placed so as to coincide with KK_1, we should have two curves passing through the arch, the neutral surface curve a_1, a_2, a_3, etc., and that of the equilibrium polygon b_1, b_2, b_3, etc.

Since both curves with their respective closing lines satisfy the conditions $\Sigma M = 0$ and $\Sigma(M.x) = 0$, then the ordinates intercepted between the two curves, or those of the imaginary type a,b, must satisfy these conditions. We have then one further condition to satisfy, *i.e.*, that $\Sigma(M.y)$ must be zero (p. 146).

Since the ordinates of the imaginary type ab must be proportional to the bending moments at the various sections, we have the condition that $\Sigma(a.b \times y)$ must be zero

$$\text{or } \Sigma(fb \times y) - \Sigma(ka \times y) = 0.$$

Therefore $\Sigma\ (fb \times y)$ must equal $\Sigma(ka \times y)$.

$\Sigma(ka \times y)$ may be written $\Sigma(y - e)\,y$, or $\Sigma y^2 - e\Sigma y$. The value of this expression is found from Table XIV.

TABLE XIV.

Length of ordinates of type y.	y^2.
$y_1 = 1{\cdot}58$	2·496
$y_2 = 3{\cdot}45$	11·903
$y_3 = 4{\cdot}15$	17·222
$y_4 = 4{\cdot}55$	20·702
$y_5 = 4{\cdot}77$	22·753
$y_6 = 5{\cdot}00$	25·000
$y_7 = 5{\cdot}10$	26·010
$y_8 = 5{\cdot}18$	26·832
$y_9 = 5{\cdot}22$	27·248
$y_{10} = 5{\cdot}23$	27·353
Sum = 44·23.	207·519 Summations for half the arch.

$$\frac{44{\cdot}2}{10} = 4{\cdot}42 = e = \text{ordinate to K K}_1 \text{ from A B.}$$

$$\begin{aligned} 2\,(\Sigma y^2 - e\Sigma y) &= 2(207{\cdot}52 - 195{\cdot}81). \\ &= 2 \times 11{\cdot}71. \\ &= 23{\cdot}42. \end{aligned}$$

And, since the neutral surface is symmetrical, these sums need only be found for half the arch, and the total multiplied by two for the whole summation.

$\Sigma(fb \times y)$ must be found for the whole curve $b_1\, b_2\, b_3 \ldots b_{20}$, but the working is simplified by proceeding in the following manner, considering the ordinates above FF_1 as positive and those below as negative. $\Sigma(f.b \times y) = [f_{10}b_{10} + f_{11}b_{11}]y_{10} + [f_9b_9 + f_{12}b_{12}]\,y_9 \ldots - [f_3b_3 + f_{18}b_{18}]y_3 - [f_2b_2 + f_{19}b_{19}]y_2 - [f_1b_1 + f_{20}b_{20}]y_1$.

If the equality $\Sigma y^2 - e\Sigma y = \Sigma(f.b \times y)$ does not hold, all the ordinates of the type $(f.b$ must be altered in the ratio of $\frac{\Sigma y^2 - \Sigma ey}{\Sigma(f.b \times y)}$. The original conditions $\Sigma M = 0$ and $\Sigma(M.x) = 0$ remain satisfied, since all the ordinates are altered in the same ratio.

Points on the true pressure curve $c_1.c_2.c_3 \ldots c_{20}$ are located on the arch ring by laying off from KK_1 ordinates Kc, for which the general equation is—

$$K.c = f.b \times \frac{\Sigma y^2 - e\Sigma y}{\Sigma(f.b \times y)}.$$

The working is shown in Tables XV. and XVI., the

TABLE XV.

1 Length of ordinates of type fb.	2 Length of ordinates of type fb.	3 Sum.	4 Length of ordinates of type y.	5 Product col. 3 × col. 4.
$f_1b_1 = -6{\cdot}72$	$f_{20}b_{20} = -5{\cdot}87$	$-12{\cdot}59$	$y_1 = 1{\cdot}58$	$-19{\cdot}892$
$f_2b_2 = -1{\cdot}80$	$f_{19}b_{19} = -1{\cdot}95$	$-3{\cdot}75$	$y_2 = 3{\cdot}45$	$-12{\cdot}937$
$f_3b_3 = -0{\cdot}25$	$f_{18}b_{18} = -0{\cdot}56$	$-0{\cdot}81$	$y_3 = 4{\cdot}15$	$-3{\cdot}362$
$f_4b_4 = +0{\cdot}57$	$f_{17}b_{17} = +0{\cdot}14$	$+0{\cdot}71$	$y_4 = 4{\cdot}55$	$+3{\cdot}230$
$f_5b_5 = +1{\cdot}07$	$f_{16}b_{16} = +0{\cdot}63$	$+1{\cdot}70$	$y_5 = 4{\cdot}77$	$+8{\cdot}109$
$f_6b_6 = +1{\cdot}35$	$f_{15}b_{15} = +0{\cdot}98$	$+2{\cdot}33$	$y_6 = 5{\cdot}00$	$+11{\cdot}650$
$f_7b_7 = +1{\cdot}53$	$f_{14}b_{14} = +1{\cdot}20$	$+2{\cdot}73$	$y_7 = 5{\cdot}10$	$+13{\cdot}923$
$f_8b_8 = +1{\cdot}63$	$f_{13}b_{13} = +1{\cdot}40$	$+3{\cdot}03$	$y_8 = 5{\cdot}18$	$+15{\cdot}695$
$f_9b_9 = +1{\cdot}71$	$f_{12}b_{12} = +1{\cdot}50$	$+3{\cdot}25$	$y_9 = 5{\cdot}22$	$+16{\cdot}965$
$f_{10}b_{10} = +1{\cdot}70$	$f_{11}b_{11} = +1{\cdot}70$	$+3{\cdot}40$	$y_{10} = 5{\cdot}23$	$+17{\cdot}782$
Summation.		$+0{\cdot}00$		$+51{\cdot}163$

$$\frac{\Sigma y^2 - \Sigma ey}{\Sigma fb \times y} = \frac{23{\cdot}42}{51{\cdot}16} = 0{\cdot}458$$

$\frac{\Sigma fb \times y}{\Sigma y^2 - \Sigma ey} = \frac{51{\cdot}16}{23{\cdot}42} = 2{\cdot}18$ and $2{\cdot}18 \times 20{\cdot}000$ = true horizontal thrust = 43·600 pounds.

distances a_1c_1, a_2c_2, a_3c_3, etc., or the arms of the bending moments at the various sections being given in Table XVI.

The tendency to tensile stress is at the intrados for a positive and at the extrados for a negative bending moment.

To draw in the curve $c_1.c_2.c_3 \ldots c_{20}$, and to find the

TABLE XVI.

Length of ordinate type *fb*.	Multiplied by.	Length of ordinate type K*c*.	Length of ordinate type K*a*.	Length of ordinate type *ac*.
f_1b_1 = − 6·72	0·458	− 3·08	− 2·85	− 0·23
f_2b_2 = − 1·80	,,	− 0·82	− 0·97	+ 0·15
f_3b_3 = − 0·25	,,	− 0·11	− 0·27	+ 0·16
f_4b_4 = + 0·57	,,	+ 0·26	+ 0·12	+ 0·14
f_5b_5 = + 1·07	,,	+ 0·49	+ 0·35	+ 0·14
f_6b_6 = + 1·35	,,	+ 0·62	+ 0·58	+ 0·04
f_7b_7 = + 1·53	,,	+ 0·70	+ 0·67	+ 0·03
f_8b_8 = + 1·63	,,	+ 0·75	+ 0·76	− 0·01
f_9b_9 = + 1·71	,,	+ 0·78	+ 0·80	− 0·02
$f_{10}b_{10}$ = + 1·70	,,	+ 0·78	+ 0·81	− 0·03
$f_{11}b_{11}$ = + 1·70	,,	+ 0·78	+ 0·81	− 0·03
$f_{12}b_{12}$ = + 1·54	,,	+ 0·70	+ 0·80	− 0·10
$f_{13}b_{13}$ = + 1·40	,,	+ 0·63	+ 0·76	− 0·13
$f_{14}b_{14}$ = + 1·20	,,	+ 0·55	+ 0·68	− 0·13
$f_{15}b_{15}$ = + 0·98	,,	+ 0·45	+ 0·58	− 0·13
$f_{16}b_{16}$ = + 0·63	,,	+ 0·29	+ 0·35	− 0·06
$f_{17}b_{17}$ = + 0·14	,,	+ 0·06	+ 0·13	− 0·07
$f_{18}b_{18}$ = − 0·56	,,	− 0·26	− 0·28	+ 0·02
$f_{19}b_{19}$ = − 1·95	,,	− 0·39	− 0·97	+ 0·08
$f_{20}b_{20}$ = − 5·87	,,	− 2·69	− 2·85	+ 0·15

magnitude of the thrusts, the true force polygon must be drawn.

From O_1 draw the line O_1H parallel to the closing line of the equilibrium polygon FF_1, to cut the force line in H, and from H draw a horizontal line H.O, making

$$HO = H_1O_1 \times \frac{\Sigma(fb \times y)}{\Sigma y^2 + e\Sigma y} = 20{,}000 \times \frac{51 \cdot 16}{23 \cdot 46} = 43{,}600 \text{ lbs.}$$

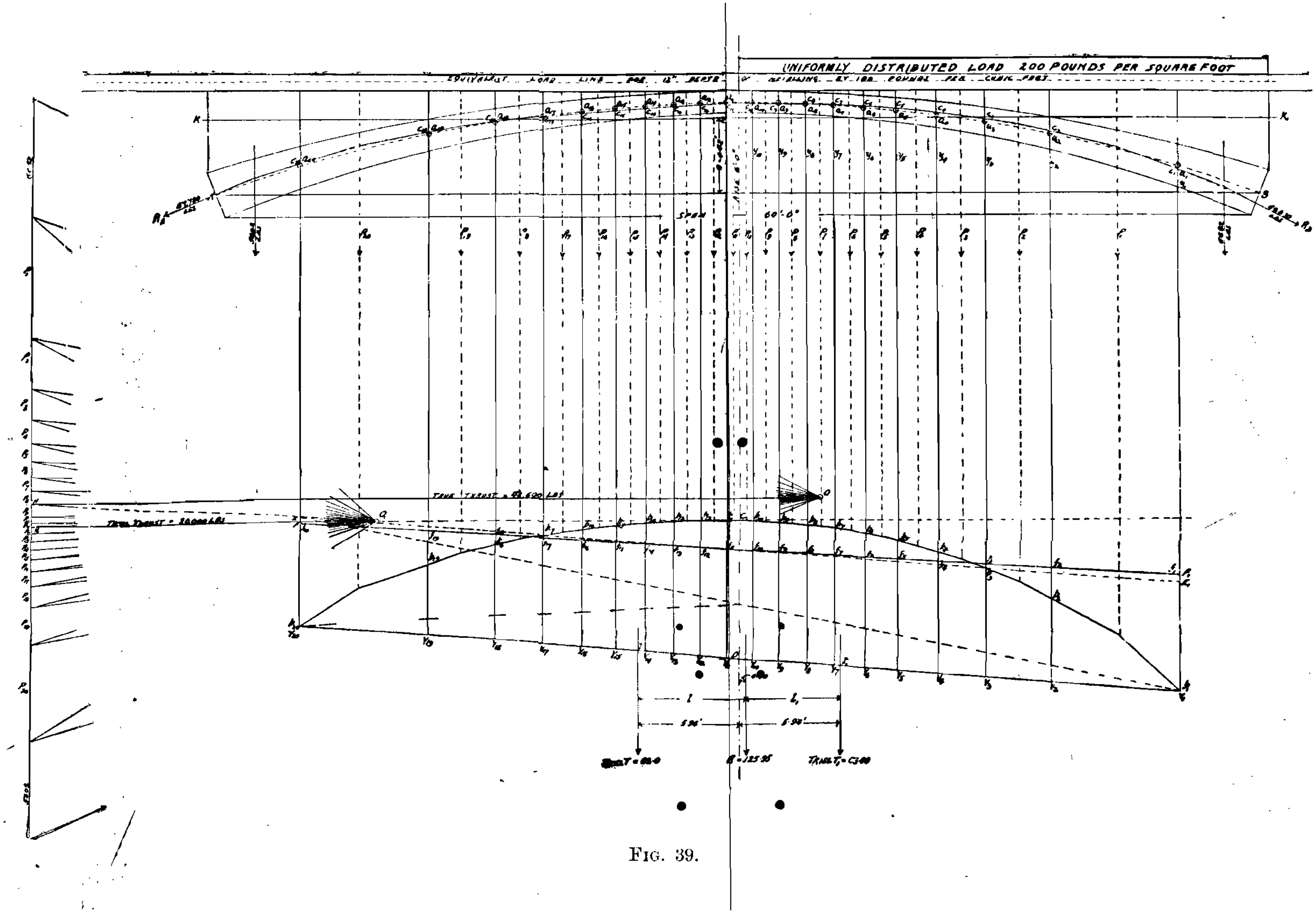
= the true horizontal thrust to the scale of loads (*vide* p. 147).

The pressure curve is now drawn in by lines parallel to the rays of the force diagram through the points $c_1.c_2.c_3 \ldots c_{20}$.

Lastly, the thrust at the springings is found by combining the loads for the end portions of the arch with the thrusts through a_1 and a_{20}. This can be done on the force diagram by setting off the respective outer loads (6,682 and 5,602) above P_1 and below P_{20} on the force line and drawing rays from their terminations to O. These outer rays will give the thrusts at the springings in magnitude and direction; consequently if we produce the pressure curve lines in the arch acting from P_1 through c_1 to the load line 6,682 and from P_{20} through c_{20} to the load line 5,602, and from the points of intersection draw lines parallel to the respective outer rays of the force diagram through the springings, we complete the pressure curve.

It must be remembered that the *bending moment* at any section is the *horizontal thrust* × *the vertical* distance between the curves $a_1.a_2.a_3 \ldots a_{20}$, and $c_1.c_2.c_3 \ldots c_{20}$, and that the *thrust* at any section must be resolved so as to act *normally* to the *radial line* at the section or since $\frac{M}{T}$ = the radial distance between the neutral surface and the pressure curve $T = M \div$ the radial distance between the neutral surface and the pressure curve (*vide* p. 141).

The working has been very fully explained for this type of arch, and the further types will be treated in less detail,



UNIFORMLY DISTRIBUTED LOAD 200 POUNDS PER SQUARE FOOT
SPAN 60'-0"

FIG. 39.

since if necessary reference can be made to this type when anything is not quite clear. When the load covers the whole span, *if the trial horizontal thrust* H_1O_1 *is drawn on the force diagram as acting between* P_{10} and P_{11}, the equilibrium curve for the loaded half of the span in the case just treated can be used, and as the two sides will be the same, this side only need be used. The line joining the ends of the equilibrium curve $b_1.b_2.b_3.$, etc., will be horizontal, and a new true closing line must be found for these conditions, and the rest of the working follows in a similar manner to that shown in the above example. *This also holds true for the cases of the Arch Hinged at the Springings and the Three-Hinged Arch.*

The thrusts and moments produced by changes of temperature must be added to those due to direct loading.

Arch Hinged at the Springings (Fig. 40).

When an arch is hinged at the springings we have the conditions that the curve of pressures must pass through the hinges and further that equation [17], p. 146, or $\Sigma(M.y) = 0$, must be satisfied.

In this case the assumed arch has the same span, rise and exterior loading as in the previous example, but a bridge with arcade spandrils has been selected,* where the positions of some of the load lines depend on the arched spandrils instead of the divisions of the arch ring.

The depth of the arch ring at the springings has been made equal to that found for the crown by Mr. Weld's formula (p. 149), the depth at the quarter points has been taken as $1\frac{1}{3}$ this amount, and the extrados curve passes

* *Vide* remarks on p. 68 as to shearing under piers of arcades.

through the points so found, a horizontal line tangent to the extrados curves on either side of the centre completing the arch ring.

The neutral surface line of the arch ring is drawn through the mid points of the depth of the arch ring, and is divided in the same manner as the previous example.

Proceeding as in the last example we obtain the values given in Table XVII.

TABLE XVII.

Division of neutral surface curve.	Length of division.	Summation.	Distance from springing to centres of division.
S_1	2·03	2·03	1·01
S_2	2·48	4·51	3·27
S_3	3·12	7·63	6·07
S_4	3·71	11·34	9·48
S_5	4·06	15·40	13·37
S_6	4·25	19·65	17·52
S_7	3·86	23·51	21·58
S_8	3·04	26·55	25·03
S_9	2·49	29·04	27·80
S_{10}	2·08	31·12	30·08

The P^s are found as follows: P_{20} is taken as the load of the roadway, spandril and arch ring from the springing to the crown of the first spandril arch.

The line of action of the load of the roadway and spandril arch will act through the centre line of the end pier, as shown by the dotted line with an arrow-head; this load must be combined with that of the arch ring to the centre of the first spandril arch, which brings the line of action to P_{20}. P_1 is found in the same manner, but the exterior load from the springing to the centre of the first

Arch Hinged at Springings.

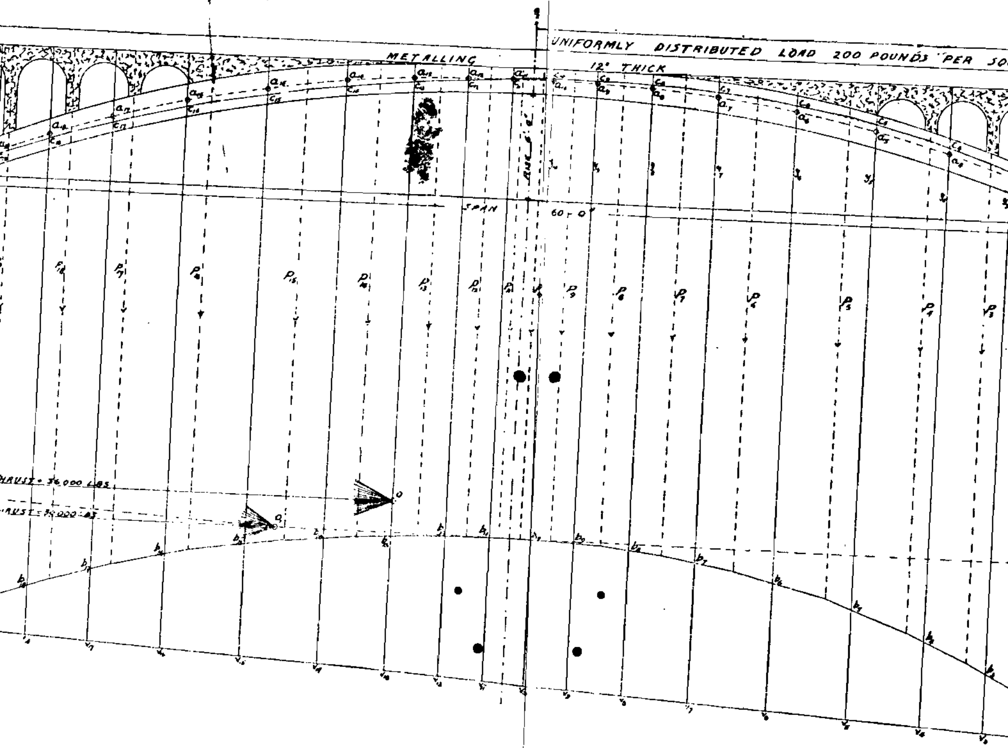


FIG. 40.

spandril arch must be added to the load acting through the centre of the first pier.

P_{19}, P_{18}, P_{17} and P_2, P_3 and P_4 act through the centres of the respective piers and the magnitude of each is that due to the whole loading from centre to centre of the respective spandril arches, including the weight of the arch ring itself.

P_{16} and P_5 are each taken as the total loads between the centre of spandril arches nearest to the centre to the vertical sections at a_{15} and a_6 and act at the centres of gravity of the respective loads.

The remainder of the P^s are taken between the vertical sections at the a^s and acting at the mid points between them, P_{11} and P_{10} being taken between a_{11} and a_{10} and the crown. The loads are given in Table XVIII., together with

TABLE XVIII.

		Summation.
P_1	1,955	1,955
P_2	2,440	4,395
P_3	2,390	6,785
P_4	2,275	9,060
P_5	3,715	12,775
P_6	2,420	15,195
P_7	1,805	17,000
P_8	1,400	18,400
P_9	1,120	19,520
P_{10}	505	20,025
P_{11}	285	20,310
P_{12}	660	20,970
P_{13}	840	21,810
P_{14}	1,125	22,935
P_{15}	1,620	24,555
P_{16}	2,655	27,210
P_{17}	1,635	28,845
P_{18}	1,755	30,600
P_{19}	1,800	32,400
P_{20}	1,455	33,855

the summation. The loads P are laid off as before on the vertical force line to the left of the diagram. A trial thrust of 30,000 lbs. is assumed in this case, acting between P_{10} and P_{11} or through the crown and the equilibrium polygon $b_1.b_2.b_3 \ldots b_{20}$ is drawn as before. The checking is accomplished in the same manner as before, the horizontal line in this case passing through P_{10} and P_{11}.

The equilibrium polygon is continued to cut the vertical lines dropped from A and B at A_1 and B_1.

Join A_1B_1.

Since the pressure curve must pass through the hinges at the springings A_1B_1 is the closing line of the equilibrium polygon, drop verticals through $a_1.a_2.a_3 \ldots a_{20}$, cutting the polygon at $b_1.b_2.b_3 \ldots b_{20}$, and the line A_1B_1 at $x_1.x_2.x_3 \ldots x_{20}$. Now suppose the line A_1B_1 to coincide with the line AB. The condition $\Sigma(M.y) = 0$ must be satisfied, but since the bending moment at any section = the horizontal thrust × the vertical distance between the two curves (*vide* p. 141), we get the condition $\Sigma[(a.b) \times y] = 0$. If we call the ordinates between AB and $a_1.a_2.a_3 \ldots a_{20}$ ordinates of the type y and $x_1b_1.x_2b_2.x_3b_3 \ldots x_{20}b_{20}$ ordinates of the type y_b.

The condition $\Sigma[(a.b) \times y] = 0$ becomes

$$\Sigma(y_b - y)y = 0.$$

The ordinates $(y_b - y)$ varying in sign according as y_b or y is the greater.

The above equation may be written—

$$\Sigma y_b y = \Sigma y^2.$$

If this equality does not hold good, all the ordinates of the type y_b must be altered in the ratio $\frac{\Sigma y^2}{\Sigma y_b y}$ to locate the true pressure curve, or generally—

$$y_c = y_b.\frac{\Sigma\, y^2}{\Sigma y_b y}.$$

By plotting the values of y_c thus found from the line AB on the verticals through $a_1.a_2.a_3.$, etc., points on the true pressure curve $c_1.c_2.c_3$ . . . c_{20} are located. The working is shown in Table XIX., together with the distances $a_1.c_1$, $a_2.c_2$, a_3c_3, etc., giving the arms of the bending moments at the various sections. The tendency to tensile stress is at the intrados for a positive, and at the extrados for a negative bending moment. To obtain the true horizontal thrust and to be able to draw the true force diagram for obtaining the thrusts and drawing in the pressure curve on the arch ring, draw from the trial pole O_1 a line O_1H parallel to A_1B_1, cutting the force line at H, and from H draw a horizontal line H.O, making $HO = H_1O_1 \times \frac{\Sigma y_b y}{\Sigma y^2}$. Then O is the true pole, and H.O measured to the scale of forces is the true horizontal thrust, in this case 36,000 lbs. Lines parallel to the rays are drawn through the points $c_1.c_2.c_3$, . . . c_{20} on the arch ring, giving the true pressure curve, and the lengths of the several rays on the force diagram measured to the scale of forces give the several thrusts. The outer rays of the force polygon give the magnitude and direction of the thrusts at the springings.

It must be remembered that the *bending moment* at any section is the *horizontal thrust* × the *vertical distance* between the curves $a_1.a_2.a_3$ . . . a_{20} and $c_1.c_2.c_3$ . . . c_{20}, and that the *thrust* at any section must be resolved so as to act

TABLE XIX.

1	2	3	4	5	6
Length of ordinates.	Length of ordinates.	Column 1 squared.	Column 1 × column 2.	y_c Col. 2 × 0·83.	Difference column 1 and column 5 giving arms of bending moments.
y_1 = 0·47	v_1b_1 = 0·56	0·22	0·26	0·47	0·00
y_2 = 1·30	v_2b_2 = 1·73	1·69	2·25	1·44	+ 0·14
y_3 = 2·27	v_3b_3 = 3·05	5·15	6·92	2·53	+ 0·26
y_4 = 3·32	v_4b_4 = 4·36	11·02	14·48	3·62	+ 0·30
y_5 = 4·30	v_5b_5 = 5·63	18·49	24·21	4·67	+ 0·37
y_6 = 5·12	v_6b_6 = 6·57	26·22	33·64	5·45	+ 0·33
y_7 = 5·69	v_7b_7 = 7·16	32·38	40·74	5·94	+ 0·25
y_8 = 6·02	v_8b_8 = 7·48	36·24	45·03	6·21	+ 0·19
y_9 = 6·10	v_9b_9 = 7·52	37·21	45·87	6·24	+ 0·14
y_{10} = 6·13	$v_{10}b_{10}$ = 7·50	37·58	45·98	6·23	+ 0·10
y_{11} = 6·13	$v_{11}b_{11}$ = 7·38		45·24	6·13	+ 0·00
		206·20			
y_{12} = 6·10	$v_{12}b_{12}$ = 7·19	× 2	43·86	5·97	− 0·13
y_{13} = 6·02	$v_{13}b_{13}$ = 6·91		41·60	5·74	− 0·28
y_{14} = 5·69	$v_{14}b_{14}$ = 6·46		36·76	5·38	− 0·31
y_{15} = 5·12	$v_{15}b_{15}$ = 5·75		29·44	4·77	− 0·35
y_{16} = 4·30	$v_{16}b_{16}$ = 4·83		20·77	4·01	− 0·29
y_{17} = 3·32	$v_{17}b_{17}$ = 2·68		12·22	3·05	− 0·27
y_{18} = 2·27	$v_{18}b_{18}$ = 2·50		5·67	2·08	− 0·19
y_{19} = 1·30	$v_{19}b_{19}$ = 1·40		1·82	1·16	− 0·14
y_{20} = 0·47	$v_{20}b_{20}$ = 0·48		0·23	0·40	− 0·07
Summation		$412·4 = \Sigma y^2$	$496·99 = \Sigma y_b y$		

$$\frac{\Sigma y^2}{\Sigma y_b y} = \frac{412·4}{497} = 0·83$$

$$\frac{\Sigma y_b y}{\Sigma y^2} = \frac{497}{412·4} = 1·2 \text{ and } 1·2 \times 30{,}000 = \textbf{True horizontal thrust}$$

$$= 36{,}000 \text{ lbs.}$$

normally to the *radial line* at that section, or since $\frac{M}{T}$ = radial distance from the neutral surface to the pressure curve T = M ÷ the radial distance between the neutral surface and the pressure curve (*vide* p. 141).

The thrusts and moments produced by changes of temperature must be added to those due to the direct loading.

Three-Hinged Arch (Fig. 41).

The following method was adopted in designing the arch ring for this type:—

The depth at the crown and springings is the same as that found for the crown of the continuous arch, *i.e.*, 15 inches, the depth at the quarter points being double this amount. The radius necessary for a segmental curve having the same rise and span as the proposed arch is found, and the centre for this curve is plotted on a vertical line through the crown. Radial lines are then drawn from this centre through the springings, and the thickness of the arch ring laid off. From the mid-point of this thickness, a distance of 2 inches was set off perpendicular to the radial line to give the centre of the hinges at the springing.

The hinge at the crown was next placed at half the depth of the arch ring, and thus three points were fixed on the neutral surface line.

The radius for a segment passing through these points was then found, and the centre from which it is struck plotted on the vertical line through the crown. Next a distance of 2 inches was laid off on the intrados and extrados on each side of the crown. The points thus found on the intrados were then joined to the springings on either side, as shown on the diagram. These lines were then bisected and lines drawn through the points thus found and the centre for the neutral surface curve. The centres for the segmental curves for the extrados and intrados must be on these lines. The arch ring being double the depth at the springings or crown at the quarter points, the mid-points on the extrados and intrados curves are laid off from the neutral surface curve, and the length of the radii were found in the usual way from the span and versed sine of these curves.

Table XX. gives the divisions of the arch ring found by the method described for the continuous arch.

TABLE XX.

Division of neutral surface curve.	Length of division.	Summation.	Distance from springings to centres of divisions giving *a*s.
S_1	1·01	1·01	0·50
S_2	1·46	2·47	1·74
S_3	2·03	4·50	3·48
S_4	3 54	8·04	6·27
S_5	5·18	13·22	10·63
S_6	7·01	20·23	16·72
S_7	5·18	25·41	22·82
S_8	3·54	28·95	27·18
S_9	2·03	30·98	29·96
S_{10}	1·46	32·44	31·71
S_{11}	1·01	33·45	32·94

This gives a division of the arch into 22 parts instead of 20, as in previous cases.

The P's are taken at the mid-points between the *a*'s excepting at the crown, where they are taken one on each

Three-Hinged Arch.

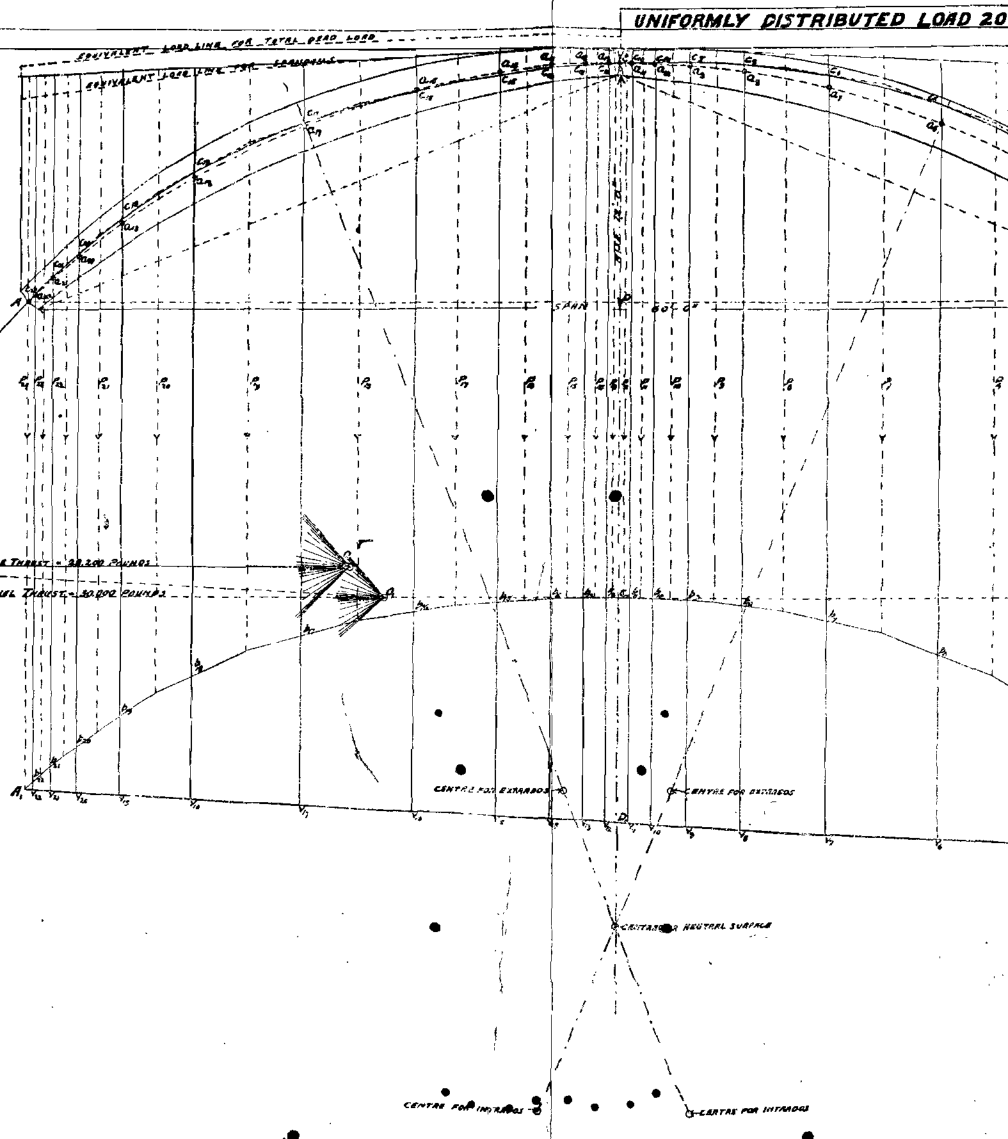


Fig. 41.

side, midway between the centre and a_{11} and a_{12}, and at the springings, where the P^s are taken midway between the outside of the spandrils and a_1 and a_{22}.

By a coincidence the lines of action of P_1 and P_{24} pass through the hinges. In this case the weight of the concrete for the arch is taken at 155 lbs. per cubic foot, that of the spandrils at 140 lbs. per cubic foot, and the weight of the filling over the spandrils and of the roadway are both taken as 100 lbs. per cubic foot.

This gives the equivalent load line as shown on the diagram, with the several loads reduced to 155 lbs. per cubic foot.

The loads P are found as described previously, the moving load on the right side being added to the similar P^s found for the left side to obtain the values of $P_1.P_2 \ldots P_{12}$.

The loads are laid off on the vertical to the left of the diagram, the force diagram and equilibrium polygon for an assumed horizontal thrust of 30,000 lbs. acting between P_{12} and P_{13} being drawn as described before. The verticals through the a^s are then dropped, giving points on the equilibrium polygon $b_1b_2b_3 \ldots b_{22}$ and $v_1v_2v_3 \ldots v_{22}$.

Now since the pressure curve on the arch ring must pass through the three hinges, all that is necessary is to scale the respective lengths of CD and C_1D_1, and if they are not the same to alter the ordinates $v_1b_1.v_2b_2.v_3b_3 \ldots v_{22}b_{22}$ in the ratio of $\frac{CD}{C_1D_1}$.

The revised lengths thus found must then be laid off from the line AB joining the hinges at the springings to give points on the true pressure curve.

TABLE XXI.

	Load lbs.	Summation lbs.
P_1	1,442	1,442
P_2	1,882	3,334
P_3	2,557	5,881
P_4	3.986	9,867
P_5	5,729	15,596
P_6	6.811	22,407
P_7	5,115	27,522
P_8	2,993	30,515
P_9	1,628	32,143
P_{10}	1,001	33,144
P_{11}	631	33.775
P_{12}	321	34.096
P_{13}	201	34,297
P_{14}	391	34,688
P_{15}	641	35,329
P_{16}	1,088	36.417
P_{17}	2,133	38,550
P_{18}	3,975	42,525
P_{19}	5,711	48.236
P_{20}	4,989	53,225
P_{21}	3,546	56,771
P_{22}	2,297	59,068
P_{23}	1,702	60,770
P_{24}	1,302	62,072

$$\frac{CD}{C_1D_1} = \frac{12 \cdot 22}{11 \cdot 48} = 1 \cdot 064.$$

$\frac{C_1D_1}{CD} = \frac{11 \cdot 48}{12 \cdot 22} = 0 \cdot 94$ and $0 \cdot 94 \times 30{,}000 = 28{,}200$ lbs. = True Horizontal Thrust.

To find the true pole and horizontal thrust for the force polygon, draw the line O_1H parallel to A_1B_1 to cut the force line at H, and from H draw a horizontal line HO, making

$$HO = H_1O_1 \times \frac{C_1D_1}{CD}.$$

Lines drawn through the points $c_1.c_2.c_3 \ldots c_{22}$ parallel to the rays $P_1P_2.P_2P_3$ . . . $P_{23}P_{24}$ will give the true pressure curve, when the sharp bends are rounded off as shown by the full line $c_1.c_2.c_3$ . . . c_{22}.

TABLE XXII.

	Length of ordinate.	Multiplied by.	Product ordinate from A B to pressure curve.
v_1b_1	0·48	1·064	0·51
v_2b_2	1·32	,,	1·40
v_3b_3	2·60	,,	2·76
v_4b_4	4·45	,,	4·72
v_5b_5	6·90	,,	7·34
v_6b_6	9·45	,,	10·04
v_7b_7	10·90	,,	11·58
v_8b_8	11·43	,,	12·16
v_9b_9	11·55	,,	12·30
$v_{10}b_{10}$	11·50	,,	12·24
$v_{11}b_{11}$	11·48	,,	12·22
$v_{12}b_{12}$	11·45	,,	12·16
$v_{13}b_{13}$	11·35	,,	12·07
$v_{14}b_{14}$	11·22	,,	11·93
$v_{15}b_{15}$	10·90	,,	11·59
$v_{16}b_{16}$	10·18	,,	10·82
$v_{17}b_{17}$	8·60	,,	9·14
$v_{18}b_{18}$	6·28	,,	6·67
$v_{19}b_{19}$	4·05	,,	4·29
$v_{20}b_{20}$	2·32	,,	2·46
$v_{21}b_{21}$	1·20	,,	1·27
$v_{22}b_{22}$	0·40	,,	0·42

The outer rays of the force diagram give the direction and magnitude of the thrusts at the springings, and the magnitude of the thrusts at the several points $c_1.c_2.c_3$ . . . c_{22} on the arch ring are given by the lengths of the rays $P_1P_2.P_2P_3.P_3P_4$ . . . $P_{23}P_{24}$ of the force diagram.

Tables XXI. and XXII. show the working.

Temperature Stresses.

It is necessary to take the effects of the rise and fall of temperature into consideration when designing an arch, as the induced stresses from these causes may be considerable.

A rise of temperature causes compressive stresses and a fall tensile stresses.

The horizontal thrust or tension due to changes of temperature always acts along the closing line of the neutral surface curve, considered as an equilibrium polygon.

In considering the effects of temperature on an arch we must assume that the whole material of the arch ring undergoes the same change. This manifestly is not likely to be correct, as the concrete being a bad conductor of heat, any particle of it will be less affected as its depth from the surface exposed to change of temperature increases.

We take, however, the average temperature of the mass, which is a very different thing from the average temperature to which it is exposed. It is very doubtful whether the range of temperature of a mass of concrete will range as much as the variation between the mean summer and mean winter temperatures. Iron and steel when used alone are employed as thin pieces, and being good conductors, respond rapidly to changes of temperature; concrete being used in thicker pieces, or in mass, and a poor conductor, responds very slowly.

An arch is usually employed for a bridge or reservoir covering, and in either case has over it a considerable thickness of earth or other equally non-conducting material; and further, it is frequently over water. Both these conditions tend to protect it against heat and cold.

Masonry bridges from time immemorial have been built without provision for temperature stresses. If a range of even 20 degrees had acted upon them, it is highly probable that many would have collapsed from this cause, and we know that they do not so collapse. In masonry bridges the spandril walls doubtless aid in resisting such effects; but reinforced concrete arches are frequently built of small rise compared with the span or with open spandrils, and consequently receive little or no assistance from the spandrils.

It appears that a liberal allowance for the range of temperature used in calculations would be 15 degrees. This would apply to most countries, the exceptions being those which have a steady high summer and steady low winter temperature, when 20 degrees might be taken. It should be remembered that the daily range is of little consequence, but that the high or low temperatures must be long continued to produce any considerable effect.

The whole question is one of great uncertainty, and must be largely a matter for the judgment of the designer.

For calculating temperature stresses it is advisable to divide the half neutral surface curve into 20 or 30 parts, such that $\frac{s}{I_c + mI_f}$ is constant. This can be done in the same manner as described on pp. 153 to 156.

If L is the span of the neutral surface curve, ϵ is the expansion or contraction due to 1 degree of temperature which may be taken as 0·000006.

t° is the greatest deviation of temperature.

H_1 is the horizontal thrust or pull due to the change of temperature.

I_c and I_f are the respective moments of inertia of the concrete and reinforcement.

E_c is the coefficient of elasticity of the concrete, which may be taken as $2{\cdot}067 \times 10^6$.

$$\text{And } m = \frac{E_f}{E_c} = 15.$$

$$\text{By the construction } \frac{I_c + mI_f}{s} \text{ is constant.}$$

As the arch may be considered symmetrical we need only consider one-half, as the stresses in the other half due to change of temperature will be identical.

We have now the condition that—

$$\frac{L\epsilon t^\circ}{2} = \Sigma.\frac{Mys}{E_c(I_c + mI_f)} \text{ (\textit{vide} equation at top of p. 146)}$$

$$\text{and } \Sigma.M = \Sigma.H_1\,(ka) \quad \therefore \frac{L\epsilon t^\circ}{2} = \Sigma.\frac{H_1(ka)\,ys}{E\,(I_c + mI_f)}$$

but H_1 and E_c are constant and the neutral surface curve is divided so that $\frac{s}{I_c + mI_f}$ is constant, consequently

$$H_1 = \frac{L\epsilon t^\circ}{2} \times \frac{E_c(I_c + mI_f)}{s} \times \frac{1}{\Sigma\,(ka)y} \quad . \quad . \quad . \quad [1]$$

The ordinates of the type $(ka) = (y - e)$.

e is found as described on p. 166, KK_1 drawn in and the y^s are scaled off the diagram at each of the a^s, all as described above, the summation $\Sigma\,(ka)y$ being taken for half the arch.

Having found H_1 acting along KK_1 the effect of this is not altered if we apply two opposite forces equal to H_1

acting at the a^s, we then have horizontal thrusts at the a^s and couples H_1 (ka) producing bending moments. The thrusts must be resolved *normally to the radial joints.* These moments and thrusts must be added to the moments and thrusts found for the direct loading.

When the arch is hinged at the springings H_1 acts along the line joining the hinges, and the value of H_1, and its moments are altered accordingly; the summation in the denominator of the last factor of equation [1] becomes therefore $\Sigma (y^2)$, the summation being taken for half the arch as before.

It must be remembered that H_1 is to be considered as a thrust and also as a pull, since, when designing an arch, it is seldom possible to tell whether it will be finished when the temperature is high or low. The moments due to temperature have a different sign as the line KK_1 is above or below the neutral surface line, in the same way as described for the moments for direct loading.

After determining the maximum bending moments and thrusts at the several sections, the method of calculation given below can be used in determining the amount of reinforcement required.

Calculations for Pieces Subjected to Bending and Direct Stresses Combined.

General Remarks.

Pieces subjected to stresses of this nature are usually arches, although other pieces may be also stressed in the same way, as, for instance, columns under eccentric loading and compression pieces under the effects of wind pressure.

There are two cases which must be considered—

1. When only one kind of stress is produced, as in the case of a column with a load bearing slightly out of the centre, or an arch in which the line of pressures lies well within the arch ring.

2. When the piece is stressed in both tension and compression.

The usual cases met with in practice are those in which the main stresses are compressive, and in the following such a disposition will be assumed.

When only Compressive Stresses are produced and the Reinforcement is of Small Sectional Area and Depth and at both sides of the Piece.

In this case we have a thrust T acting at the neutral surface and a bending moment M. Under the effect of these the section A B (Fig. 42) may be supposed to take up position $A_2 B_2$, while A A_1 and B B_1 represent the minimum and maximum compressive stresses in the concrete c_1 and c_2.

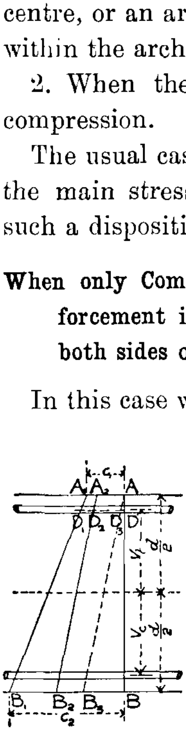


FIG. 42.

As seen from Fig. 42—

The thrust $T = db\left\{c_1 + \frac{1}{2}(c_2 - c_1)\right\} + f_c\,\omega_c + f_t\,\omega_t$

$$= \frac{db}{2}(c_2 + c_1) + f_c\,\omega_c + f_t\,\omega_t. \qquad [1]$$

The bending moment

$$M = \frac{1}{2}(c_2 - c_1)\,d \times \left(\frac{1}{2} - \frac{1}{3}\right) d \times b + f_c\,\omega_c\,v_c - f_t\,\omega^t\,v_t.$$

$$= \frac{1}{12}(c_2 - c_1)\,d^2 b + f_c\,\omega_c\,v_c - f_t\,\omega_t\,v_t. \qquad [2]$$

To find the values of f_c and f_t, it will be seen from Fig. 42 that the hypothesis of the conservation of plane sections gives us—

$$\text{Since } DD_3 = \frac{BB_3 \times AD}{AB}$$

$$DD_2 = AA_2 + \frac{BB_3 \times AD}{AB}$$

$$\text{but } DD_2 = \frac{f_t}{E_f},\ AA_2 = \frac{c_1}{E_c} \text{ and } BB_3 = \frac{c_2 - c_1}{E_c},$$

$$\text{also } AD = \frac{d}{2} - v_t \text{ and } AB = d.$$

We have then

$$f_t = m\left\{\frac{(c_2 + c_1)}{2} - (c_2 - c_1)\frac{v_t}{d}\right\}, \quad . \quad . \quad [3]$$

and similarly—

$$f_c = m\left\{\frac{(c_2 + c_1)}{2} + (c_2 - c_1)\frac{v_c}{d}\right\}. \quad . \quad . \quad [4]$$

Substituting [3] and [4] in [1] and [2] we get—

$$T = \frac{db}{2}(c_2 + c_1) + m\left\{\frac{(c_2 + c_1)}{2} + (c_2 - c_1)\frac{v_c}{d}\right\}\omega_c$$
$$+ m\left\{\frac{(c_2 + c_1)}{2} - (c_2 - c_1)\frac{v_t}{d}\right\}\omega_t$$

$$T = \frac{c_2 + c_1}{2}\left\{db + m(\omega_c + \omega_t)\right\} + (c_2 - c_1)\frac{m}{d}\left\{\omega_c v_c - \omega_t v_t\right\}, \quad . \quad . \quad . \quad . \quad [5]$$

and—

$$M = \frac{1}{12}(c_2 - c_1)d^2 b + m\left\{\frac{(c_2 + c_1)}{2} + (c_2 - c_1)\frac{v_c}{d}\right\} w_c v_c$$
$$- m\left\{\frac{(c_2 + c_1)}{2} - (c_2 - c_1)\frac{v_t}{d}\right\}\omega_t v_t$$

$$M = \frac{(c_2 - c_1)}{d}\left\{\frac{1}{12}d^3b + m(\omega_c v_c^2 + \omega_t v_t^2)\right\} + \frac{(c_2 + c_1)}{2}m(\omega_c v_c - \omega_t v_t). \quad . \quad . \quad . \quad . \quad [6]$$

When $c_1 = 0$, *i.e.* at the limit when the whole piece is in compression and when $v_c = v_t$—

$$T = c_2\left[\left\{\frac{db + m(\omega_c + \omega_t)}{2}\right\} + \frac{mv}{d}(\omega_c - \omega_t)\right],$$

and—

$$M = c_2\left[\left\{\frac{d^3b + 12mv^2(\omega_c + \omega_t)}{12d}\right\} + \frac{mv}{2}(\omega_c - \omega_t)\right],$$

$$\text{and } M = \frac{1}{6}\cdot\frac{d^3b + 12mv^2(\omega_c + \omega_t) + 6dmv(\omega_c - \omega_t)}{d^2b + md(\omega_c + \omega_t) + 2mv(\omega_c - \omega_t)}. \quad [7]$$

If v_t and v_c are not equal the equation [7] must be altered by bringing v_t and v_c inside the brackets.

If the reinforcement is symmetrical $\omega_c = \omega_t$ and $v_c = v_t$ and we get—

$$T = \frac{c_2 + c_1}{2}(db + 2m\omega), \quad . \quad . \quad [8]$$

$$\text{and } M = \frac{c_2 - c_1}{d}\left(\frac{d^3b}{12} + 2m\omega v^2\right), \quad . \quad . \quad [9]$$

and when $c_1 = o$

$$\frac{M}{T} = \frac{1}{6d}\cdot\frac{d^3b + 24m\omega v^2}{db + 2m\omega}. \quad . \quad . \quad [10]$$

When Tensile Stresses are induced, the Reinforcement being of small Sectional Area and Depths and at both sides of the Piece.

When this disposition occurs, we shall have (Fig. 43)—

$$T = \frac{1}{2}cub + \omega_c f_c - \omega_t f_t, \quad . \quad . \quad [11]$$

$$M = \frac{1}{2}cub\left(\frac{d}{2} - \frac{1}{3}u\right) + f_c\omega_c v_c + f_t\omega_t v_t, \quad . \quad [12]$$

and, as in the case of piece subjected to simple bending—

$$f_c = cm \frac{(u - a)}{u}, \quad . \quad . \quad . \quad [13]$$

$$f_t = cm \frac{\{d - (u + \beta)\}}{u}. \quad . \quad . \quad [14]$$

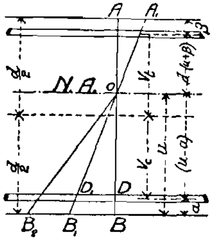


FIG. 43.

Substituting [13] and [14] in [11] and [12]—

$$T = \frac{c}{u}\left[\frac{1}{2}u^2b + m\{\omega_c(u - a) - \omega_t[d - (u + \beta)]\}\right], \quad . \quad . \quad [15]$$

$$M = \frac{c}{u}\left[\frac{1}{2}u^2b\left(\frac{d}{2} - \frac{1}{3}u\right) + m\{\omega_c(u - a)v_c + \omega_t[d - (u + \beta)]v_t\}\right], \quad . \quad . \quad . \quad [16]$$

with a symmetrical reinforcement—

$$\omega_c = \omega_t \text{ and } v_c = v_t.$$

We get therefore—

$$T = \frac{c}{u}\left[\frac{1}{2}u^2b + m\,\omega\{(u - a) - [d - (u + \beta)]\}\right]. \quad [17]$$

And as $(u - a) + \{d - (u + \beta)\} = d - (a + \beta) = 2v$

$$M = \frac{c}{u}\left\{\frac{1}{2}u^2b\left(\frac{d}{2} - \frac{1}{3}u\right) + 2\,m\,\omega v^2\right\}. \quad . \quad [18]$$

When the Reinforcement is only on one Side.

If the curve of pressures remains always on one side of the neutral surface of an arch, one reinforcement may be sufficient.

We have then for the case where the whole arch is in compression, by retaining the reinforcement under the greatest compression and eliminating that which tends towards tension—

$$T = \frac{c_2 + c_1}{2} (db + m\omega_c) + (c_2 - c_1) \frac{m}{d} \omega_c v_c, \quad . \quad [19]$$

$$M = \frac{(c_2 - c_1)}{d} \left\{ \frac{1}{12} d^3 b + m\omega_c v_c^2 \right\} + \frac{c_2 + c_1}{2} m \omega_c v_c, \quad . \quad [20]$$

$$\text{and } f = m \left\{ \frac{(c_2 + c_1)}{2} + (c_2 - c_1) \frac{v_c}{d} \right\}. \quad . \quad [21]$$

And for the case when part of the arch is in tension, retaining the reinforcement on this side—

$$T = \frac{c}{u} \left\{ \frac{1}{2} u^2 b - \omega_t v_t \right\}, \quad . \quad . \quad . \quad [22]$$

$$M = \frac{c}{u} \left\{ \frac{1}{2} u^2 b \left(\frac{d}{2} - \frac{1}{3} u \right) + m \omega_t \{ d - (u + \beta) \} v_t \right\}, \quad [23]$$

$$\text{and } f = cm \frac{\{ d - (u + \beta) \}}{u}. \quad . \quad . \quad . \quad [24]$$

In all these formulæ the curve of pressures is supposed to be passing below the neutral surface of the piece as shown in the figures, but the same formulæ will of course apply if it passes above, the piece being considered as reversed.

Use of the above Formulæ.

When the reinforcement is symmetrical, and we require to use these formulæ *to check a piece of given dimensions*, we proceed as follows :—

Taking the value of m as 15 we first try whether the piece is subjected to tension, using equation [10], which becomes—

$$\frac{M}{T} = \frac{1}{6d} \cdot \frac{d^3 b + 360 \omega v^2}{db + 30\omega}. \quad . \quad . \quad . \quad [25]$$

If $\frac{M}{T}$ is equal or less than the value in equation [25], we know that there will be no tensile stresses, and we proceed to find the values of c and f under these conditions.

We have from [8] and [9]—

$$c_2 + c_1 = \frac{2T}{db + 30\omega}$$

$$\text{and } c_2 - c_1 = \frac{12Md}{d^3b + 360\omega v^2}.$$

From which we get—

$$c_2 = \frac{T}{db + 30\omega} + \frac{6Md}{d^3b + 360\omega v^2}, \quad . \quad . \quad [26]$$

and—

$$c_1 = \frac{T}{db + 30\omega} - \frac{6Md}{d^3b + 360\omega v^2}. \quad . \quad [27]$$

By inserting the values thus found for c_2 and c_1 in [3] and [4], and as the reinforcement is symmetrical, replacing v_t and v_c by v, we find the values of f_t and f_c.

If we find from equation [25] that tensile stresses will be produced, we must find the value of u.

From equation [15] we get, since the reinforcements are symmetrical—

$$c = \frac{Tu}{\frac{1}{2}u^2b + 15\omega\{(u - a) - [d - (u + \beta)]\}}, \quad [28]$$

and from equation [16]—

$$c = \frac{Mu}{\frac{1}{2}u^2b\left(\frac{d}{2} - \frac{1}{3}u\right) + 15\omega v\{(u - a) + [d - (u + \beta)]\}} \quad [29]$$

and since these two values of c must equal one another, we get—

$$\frac{T}{\frac{1}{2}u^2b + 15\omega[(u - a) - \{d - (u + \beta)\}]}$$

$$= \frac{M}{\frac{1}{2}u^2b\left(\frac{d}{2} - \frac{1}{3}u\right) + 15\omega v[(u - a) + \{d - (u + \beta)\}]}. \quad [30]$$

Substituting the various values we get an equation from which we can find u.

This equation will be most readily solved (since it contains the cubes of u) by inserting several values of u and obtaining several values for the two sides of the equation and plotting these on squared paper to find the correct value of u.

Having obtained u the value of c is deduced from equations [28] or [29], and the values of f_t and f_c follow from equations [13] and [14], substituting 15 for m.

The values found for c_2 or c should not exceed 500 lbs. per square inch; unless M is very large compared with T (a condition which possibly may occur in some pieces such as roof rafters without ties), when c may be taken as 600 lbs. per square inch.

If it is required to determine the dimensions for a piece having only the values of M and T, we may either assume values for d and for β and a (which have the same value), or assume a value for β and also the percentage of reinforcement or $\psi = \frac{2\omega}{bd}$ Where 2ω is the *whole area* of the two reinforcements and d is the *whole depth* of the piece. A symmetrical reinforcement is usually employed for pieces of this kind.

The value of β may always be assumed to be 2 inches, if the reinforcement is of small depth compared with that of the piece, small bars being used and the requisite sectional area being made up by the closer spacing of small section bars as the total area required increases. If however it is thought desirable in the designer's judgment to alter this dimension, the following equations can easily be altered accordingly.

If we assume a depth d and a value for β,

Then $\nu = \frac{d}{2} - \beta$, and if $\beta = 2$, then $\nu = \frac{d-4}{2}$.

ω may be found direct from equation [26] by substituting for c_2 its maximum value (500 lbs. per square inch).

The values thus obtained must satisfy the expression from equation [25].

$$\frac{M}{T} \overline{\leq} \frac{1}{6d} \cdot \frac{d^3b + 360\omega\nu^2}{db + 30\omega}. \quad . \quad . \quad [31]$$

If we assume a value for ψ we have—

$$\psi = \frac{2\omega}{bd} \text{ and } \nu = \frac{d-4}{2}.$$

Equation [26] becomes by inserting these values—

$$c_2 = \frac{T}{db + 15bd\psi} - \frac{6Md}{d^3b + 45bd\psi(d-4)^2}$$

$$\text{or } bc_2 = \frac{T}{d(1 + 15\psi)} - \frac{6M}{d^2 + 45\psi(d-4)^2}, \quad [32]$$

giving c_2 its maximum value (500 lbs. per square inch), and b its value (usually 12 inches), and inserting the assumed value for ψ we can solve equation [32] for d by trying several values for d, and plotting the values for each side of the equation on squared paper, and finding the intersection of the respective curves.

Having obtained the value of d, ω is obtained from the equation $\omega = \frac{\psi bd}{2}$.

The values found must satisfy the expression [31].

If in either of these cases the condition [31] is not satisfied, we may reduce the value assumed for ψ or increase the depth d until the proper values are found, or we must consider the case where tensile stresses exist.

When Tensile Stresses exist.

If we assume a value for d and β, we get from equations [17] and [18]—

$$T = \frac{c}{u}\left[\frac{1}{2}u^2b + m\omega\{(u - a) - [d - (u + \beta)]\}\right]$$

and—

$$M = \frac{c}{u}\left[\frac{1}{2}u^2b\left(\frac{d}{2} - \frac{u}{3}\right) + 2m\omega v^2\right].$$

Assuming as before that $\beta = 2$ inches, we have also $m = 15$, $(u - a) = (u - 2)$ and $[d - (u + \beta)] = [d - (u + 2)]$, and $v = \frac{d}{2} - \beta = \frac{d - 4}{2}$, and giving c its maximum value (500 lbs. per square inch).

These values inserted in the above equations give—

$$\frac{Tu}{c} = \frac{1}{2}u^2b + 15\omega(2u - d),$$

$$\text{or } \omega = u.\frac{\dfrac{T}{500} - \dfrac{1}{2}ub}{15(2u - d)} \quad . \quad . \quad . \quad [33]$$

and—

$$\frac{Mu}{c} = \frac{1}{2}\left[u^2b\left(\frac{d}{2} - \frac{u}{3}\right) + 15\omega(d - 4)^2\right]$$

or—

$$\omega = u\ .\frac{\dfrac{M}{250} - ub\left(\dfrac{d}{2} - \dfrac{u}{3}\right)}{15(d - 4)^2}. \quad . \quad . \quad [34]$$

The values of ω from [33] and [34] must be the same. We have therefore—

$$\frac{\dfrac{T}{250} - ub}{(2u - d)} = \frac{\dfrac{M}{125} - \dfrac{ub}{3}(3d - 2u)}{(d - 4)^2}. \quad . \quad [35]$$

We know T, M, d and b (usually 12 inches), and therefore we can find the value of u from [35] by trying various values for u and plotting each side of the equation on squared paper as before described. When the value of u has been determined, ω is found from [33] or [34], and f_c and f_t from [13] and [14] by inserting the values $(u - a) = u - 2$, and $[d - (u + \beta)] = [d - (u + 2)]$.

If we assume a value for ψ and β, we write as before,
$\psi = \frac{2\omega}{bd}$ or $\omega = \frac{\psi bd}{2}$.

Substituting this value of ω in [33] and [34] we obtain—

$$\psi = \frac{u}{b} \cdot \frac{\frac{T}{250} - ub}{15(2u - d)} \quad . \quad . \quad . \quad [36]$$

and—

$$\psi = \frac{u}{b} \cdot \frac{\frac{M}{125} - \frac{ub}{3}(3d - 2u)}{15d(d - 4)^2} \quad . \quad . \quad [37]$$

We know T, M, ψ and b (usually 12 inches), and can therefore obtain from [36] and [37] two equations containing u and d.

By trying various ratios of u in respect to d we find values of d of each of the two equations, and plotting these on squared paper we find the true value of d at the intersection of the curves for the respective equations, and the true value of d inserted in either of the equations [36] and [37], will give u.

ω follows from the equation $\omega = \frac{\psi bd}{2}$, and f_c and f_t are found as before from equations [13] and [14]. If any other value than 500 is assumed for c the **equations** [35], [36] and [37] must be altered accordingly.

PIECES WITH REINFORCEMENTS OF LARGE SECTIONAL AREA.

When reinforcements of large sectional area are employed the reinforcements are always symmetrical. In this case, however, both the depth and the sectional area of the reinforcements must be taken into account, and the resistance of the concrete which is replaced by the reinforcements when in compression must be deducted. As before, (f_c) and (f_t) will be taken as the mean resistance of the reinforcing sections, and (f_{cm}) and (f_{tm}) as their maximum resistance, (i) representing their moment of inertia about their centre of gravity.

When only Compressive Stresses are Produced (Fig. 44).

We get for the mean stress on the area of concrete replaced by the reinforcement under greatest compression—

$$c_1 + (c_2 - c_1)\frac{\frac{d}{2} + v}{d}$$

and for the area of concrete replaced by the reinforcement under least compression—

$$c_1 + (c_2 - c_1)\frac{\frac{d}{2} - v}{d}$$

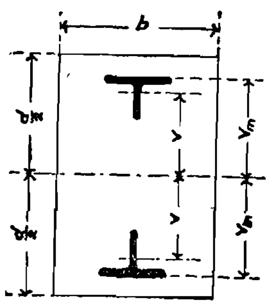


FIG. 44.

Consequently similarly to equation [1], p. 186—

$$T = \frac{1}{2}(c_2 + c_1)db - \omega\left(2c_1 + \frac{(c_2 - c_1)}{2d}[(d + 2v) + (d - 2v)]\right) + \omega(f_c + f_t)$$

$$T = \frac{1}{2}(c_2 + c_1)\,(db - 2\omega) + \omega(f_c + f_t), \quad . \quad . \quad . \quad [38]$$

and similarly to equation [2], p. 186—

$$M = \frac{1}{12}(c_2 - c_1)d^2b - \omega v\left\{c_1 + (c_2 - c_1)\frac{(d + 2v)}{2d} - c_1 - (c_2 - c_1)\frac{(d - 2v)}{2d}\right\} - \frac{(c_2 - c_1)}{d}(i_c + i_t) + \omega v(f_c - f_t)$$

$$+ \frac{i}{v}(f_c - f_t). \quad . \quad . \quad . \quad [39]$$

$$M = (c_2 - c_1)\left\{\frac{1}{12}d^2b - \frac{2v^2\omega}{d} - \frac{2i}{d}\right\} + \omega v(f_c - f_t) + \frac{i}{v}(f_c + f_t).$$

And we have also—

$$f_t = m\left\{\frac{(c_2 + c_1)}{2} - (c_2 - c_1)\frac{v}{d}\right\}, \quad . \quad . \quad [40]$$

$$f_c = m\left\{\frac{(c_2 + c_1)}{2} + (c_2 - c_1)\frac{v}{d}\right\}, \quad . \quad . \quad [41]$$

$$f_t = m\left\{\frac{(c_2 + c_1)}{2} - (c_2 - c_1)\frac{v_m}{d}\right\}, \quad . \quad . \quad [42]$$

$$f_c = m\left\{\frac{(c_2 + c_1)}{2} + (c_2 - c_1)\frac{v_m}{d}\right\}. \quad . \quad . \quad [43]$$

Inserting the values of [40] and [41] in [38] and [39] we get—

$$T = (c_2 + c_1)\left\{\left(\frac{db}{2} - \omega\right) + m\omega\right\}$$

$$T = (c_2 + c_1)\left\{\frac{db}{2} + \omega(m - 1)\right\}, \quad . \quad . \quad [44]$$

and—

$$M = (c_2 - c_1)\left\{\frac{1}{12}d^2b - \frac{2v^2\omega}{d} - \frac{2i}{d}\right\} + \frac{2v\omega^2 m}{d}(c_2 - c_1)$$

$$+ \frac{2mi}{d}(c_2 - c_1),$$

$$M = \frac{(c_2 - c_1)}{d}\left\{\frac{1}{12}d^3b - 2\omega v^2(m - 1) + 2i(m - 1)\right\}$$

$$M = \frac{(c_2 - c_1)}{d}\left\{\frac{1}{12}d^3b + 2(m - 1)\,(i - \omega v^2)\right\}. \quad . \quad [45]$$

When Tensile Stresses are Produced (Fig. 45).

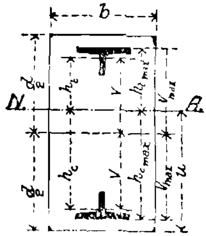


FIG. 45.

The stress on the area of concrete replaced by the compressive reinforcement will be

$$c\frac{h_c}{u}$$

and similarly to equation [11], p. 188.

$$T = \frac{1}{2}cub - c\frac{h_c}{u}\omega + \omega(f_c - f_t)$$

$$T = \frac{c}{u}\left(\frac{1}{2}u^2b - h_c\omega\right) + \omega(f_c - f_t). \quad [46]$$

and similarly to equation [12], p. 188—

$$M = \frac{1}{3}cub\left(\frac{d}{2} - \frac{u}{3}\right) - c\frac{h_c}{u}\omega v - \frac{c}{u}i_c + \omega v(f_c + f_t)$$

$$\times\ i\left(\frac{f_c}{h_c} + \frac{f_t}{h_t}\right), \quad . \quad . \quad . \quad [47]$$

and further we have—

$$f_t = cm\frac{(h_t b)}{u}, \quad . \quad . \quad . \quad [48]$$

$$f_c = cm\frac{(h_c u)}{u}, \quad . \quad . \quad . \quad [49]$$

$$f_{tm} = cm\frac{(h_{tm} b)}{u}, \quad . \quad . \quad . \quad [50]$$

$$f_{cm} = cm\frac{(h_{cm} u)}{u}. \quad . \quad . \quad . \quad [51]$$

Substituting these values from [48] and [49] in [46] and [47] we get—

$$T = \frac{c}{u}\left(\frac{1}{2}u^2b - h_c\omega\right) + \frac{cm\omega}{u}(h_c - h_t),$$

$$T = \frac{c}{u}\left[\frac{1}{2}u^2b + \omega h_c(m - 1) - m\omega h_t\right], \quad . \quad [52]$$

and—

$$M = \frac{c}{u}\left\{\frac{1}{3}u^2b\left(\frac{d}{2} - \frac{u}{3}\right) - h_c\omega v - i\right\} + \frac{\omega vcm}{u}(h_c + h_t) + \frac{2icm}{u}$$

$$M = \frac{c}{u}\left[\frac{1}{3}u^2b\left(\frac{d}{2} - \frac{u}{3}\right) - h_c\omega v - i + \omega vm(h_c + h_t) + 2im\right]$$

$$M = \frac{c}{u}\left[\frac{1}{3}u^2b\left(\frac{d}{2}-\frac{u}{3}\right) + \omega rh_c(m-1) + i(2m-1) + m\omega rh_t\right]. \quad . \quad . \quad . \quad . \quad [53]$$

SPHERICAL AND CONICAL COVERINGS.*

The graphical method of treatment for obtaining the stress of domes of thin shells, given below, is very simple and direct.

A, Fig. 47, p. 204, shows the half-section of a thin dome of uniform thickness and of uniform weight per square foot, which is hemispherical and of a material capable of resisting tensile and compressive stresses.

Assume the vertical centre line as representing to some scale yet to be determined the weight of the dome, or of some section of it. Divide this line into any number of convenient parts (for convenience, sixteen equal parts have been taken), and mark the divisions $1_{\prime}$, $2_{\prime}$, $3_{\prime}$. Draw through $1_{\prime}$, $2_{\prime}$, $3_{\prime}$, etc., horizontal lines to cut the section of the dome in 1, 2, 3, etc.

Then as the area of any segment of a sphere equals $c\,h$, where c is the circumference of the sphere and h is the height of the segment, any length such as $\overline{2_{\prime}, 3_{\prime}}$, or $\overline{5_{\prime}, 6_{\prime}}$, on the centre vertical, measured on the same scale upon which that vertical represents the whole weight of the dome or the chosen portion of it, will give the weight of the segment, $\overline{2, 3}$, or $\overline{5, 6}$, of the dome or chosen portion.

As the thickness of the dome is inconsiderable, the pressure may be considered as uniformly distributed over any section such as 1, and therefore tangential to the surface. Draw through 1 a line $\overline{1, a}$ tangential to the

* The graphical method here given is based on a paper by W. Dunn, F.R.I.B.A., published in the "Transactions of the Royal Institute of British Architects," March, 1904, and is reproduced from "Reinforced Concrete," 3rd edition, by Marsh & Dunn.

surface (at right angles to the radius $\overline{1_{,}\ 16_{,,}}$), and through o' produce ao' indefinitely. Through o' and $1_{,}$ draw $\overline{o't^1}$ and $\overline{1t^1}$ to intersect in t^1, completing the triangle of forces holding that part $o'1$ of the dome in equilibrium; $\overline{o'1_{,}}$ is the weight of it (being the load on point 1), the line $\overline{1t^1}$ the direct compression uniformly distributed over the horizontal section, and the line $\overline{o't^1}$ the radial thrust, all measured to the same scale (not yet determined) as $o'16'$.

If the point $1_{,}$ were taken very near the crown, the load would be very small, and the horizontal thrust and direct stress would also be reduced; at the crown itself there is no stress. This is one of the material points of difference between the dome and the arch, which latter construction has always a thrust at the crown.

Proceeding to section 2, we form the stress diagram giving the four-sided figure $\overline{1_{,}\ 2_{,}}\ t^2t^1$; $1_{,}\ 2_{,}$ being the weight of the segment, $\overline{2_{,}t^2}$ the direct stress upon the lower section, $\overline{t^2t^1}$ giving the radial thrust. Proceeding similarly for the remainder of the figure, the direct thrusts cut the horizontal lines further and further from the load line until we reach $6_{,}$, when the intersections begin to approach the load line again, making the polygon of forces for each section similar to the four-sided figure *abcd* (Fig. 46).

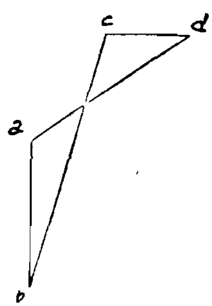


FIG. 46.

The diagram of stresses may take another form (B,

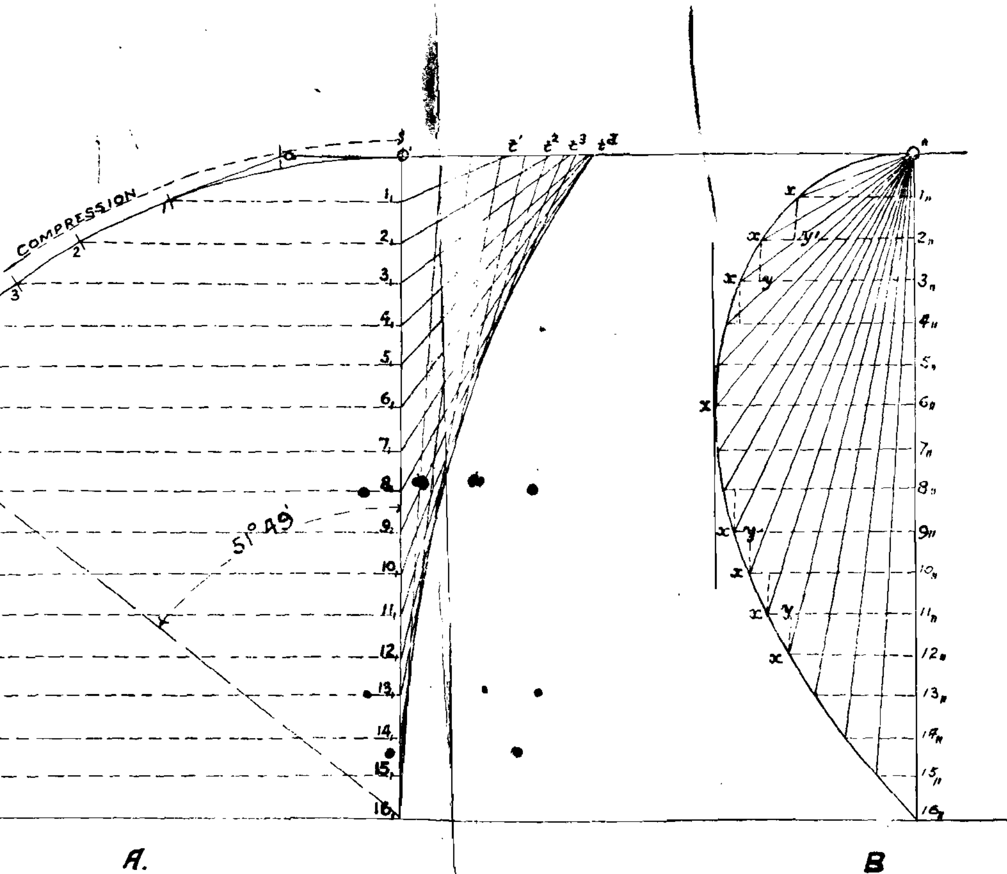


Fig. 47.

(Fig. 47). Set off the load line as before, and through o'' draw lines parallel to those tangential to the surface of the dome at the various points to intersect with the horizontal lines. Through these intersections draw the curve from o'' to $16_{''}$. Then we have the stress diagram but of form x, y, x, o'' following the vertical, then the horizontal, and then the inclined line; the radial thrust above the point where the curve turns to the load line again and the radial tensions below that point being given by the differences $\overline{x,y}$.

The lengths $x1''$ correspond to $o_{,}\ t'$ in the first figure, and similarly the $\overline{x,y\text{'s}}$ correspond to the lengths $t^1\ t^2$, $t^2\ t^3$, etc.

These thrusts and tensions, or the $\overline{x,y\text{'s}}$, must be resolved into equivalent thrusts or tensions acting at right angles to the plane of the paper—that is, actually upon the vertical faces of the section.

Let $o''16_{''}$ represent the total weight of the dome, then any $\overline{x,y}$ shows the total radial thrust upon the corresponding section, which we shall call R. Being uniformly distributed its intensity per unit of circumference equals

$$\frac{\text{R}}{\text{units in the circumference}},$$

just as the intensity of pressure on a column equals the total pressure divided by the units of area in the column.

In any circular ring under uniform normal pressure (C, Fig. 47), as in a cylinder holding water, the resultant tension or compression T (which we call hoop tension) equals the intensity of the radial pressure multiplied by the radius, that is—

$$\frac{\text{R} \times \text{radius}}{\text{circumference}} = \text{hoop thrust or tension} = \text{T}.$$

This $\frac{\text{radius}}{\text{circumference}}$ is a constant quantity for any circle, and equals $\frac{1}{6{\cdot}2832}$ nearly, so that $\frac{R}{6{\cdot}2832} = T$.

When, therefore, $o''16_{''}$ represents the total weight of the dome, we must divide each horizontal $\overline{x,y}$ by 6·2832 for the hoop tension.

Suppose we take $\overline{o''16_{''}}$ to represent $\frac{1}{6{\cdot}2832}$ of the weight of the dome, then we shall not require to divide the $\overline{x,y's}$, as they will each equal the hoop tension at that point; *i.e.*, if we take, not the weight of 360° but $\frac{360}{6{\cdot}2832}$, or 57·3 of the dome, the horizontal $\overline{x,y's}$ give the hoop tension or compression directly.

Form such a scale that $\overline{o''16_{''}}$ measures the weight of 57·3°; the lengths $\overline{xo''}$ measured to that scale give the total compression on a horizontal plane on a segment of 57·3° of the dome (this segment is in plan).

As the length of an arc of 57·3° equals its radius we have only to divide the lengths $\overline{x,o''}$ by the radius at the corresponding points to get the pressure per lineal unit on the horizontal section.

At the joint where the hoop thrust changes to hoop tension the maximum horizontal thrust is caused as is clearly seen in B, Fig. 47; it is the sum of all the thrusts xy above it. Below that joint this thrust is diminished by the tensions xy in each ring, until in the hemispherical dome these exactly balance the sum of the thrusts xy. If the dome were to spring from the joint where the hoop thrust changes to hoop tension, the provision to prevent the supports spreading would be the maximum obtainable for the dome under consideration. This joint is frequently called the joint of rupture; it is situated at a height above the springing line of $\frac{1}{2}(\sqrt{5\text{-}1})$ radius, or about 51°·49′ from the vertical. Above that joint the dome tends to collapse inwards; below it tends to spread outwards.

If the dome is segmental, springing from, say, the level 6, $6_{\prime}$, the stress diagram at the point 6 is found by drawing a vertical line representing the weight of the dome $o'6_{\prime}$ (A, Fig. 47). Through the lower end of this line, draw $6_{\prime}t^6$ parallel to the tangent at the springing and through the upper end a horizontal line o^1t^6. Then $o't^6$ gives the total radial thrust at the springing and $6_{\prime}t^6$ the total compression *on the horizontal plane at the same point; or, if $o'6_{\prime}$* represents

$$\frac{\text{the weight of the dome}}{6\cdot2832}$$

then $o_{\prime}t^6$ will equal the hoop tension at the base. This would have to be resisted by a ring or by abutments.

The weight of the dome or its covering may be estimated in the following manner. Above any horizontal section the weight of the dome will be $w \times 2\,\pi\,r\,v$, where r is the radius to which the dome is struck, v the rise, and w the weight per unit area, the radius being measured to the centre of the section.

The same methods can be used for conical coverings, since it happens that the weights above any horizontal section are proportional to the vertical distances from the apex of the cone to that section.

In a cone the greatest horizontal thrust is at the base if the abutments do not yield, or if there is a reinforcement to tie in the base. If the abutments yield, the cone itself, if capable of doing so, also supplies the necessary resistance to twisting open.

These results are only correct for *thin* coverings of material capable of resisting tension and compression, of true spherical or conical section, and of uniform weight per unit of service. Such a material as reinforced concrete may be assumed to satisfy these conditions.

If we put an eye to the dome (*i.e.* omit the central upper part, say the part above the line $\overline{1.1_{\prime}}$), then the horizontal line o', t^1, t^2, t^3, etc. is lowered to the level of $\overline{1.1_{\prime\prime}}$ and the position of the joint of rupture is also lowered. If a heavy load, such as a lantern, were put at 1, then the horizontal line o', t^1, t^2, t^3, etc., would be raised, increasing the part under hoop tension and diminishing the part under hoop compression. If the section is varied and becomes pointed, or of any other curvature, there is also a change in the position of this joint.

The intensity of the stresses in spherical domes as found by such a diagram as A or B, Fig. 47, follows from the weight of the dome, which varies directly with the thickness, supposed to be uniform. As the area to resist the resulting stresses is correspondingly varied, it appears that whatever the thickness in a hemispherical dome of uniform thickness, the unit stresses theoretically are always the same for the same radius. We say theoretically, as the foregoing theory does not take account of the elastic yielding of the dome which may cause the centre of pressure to leave the centre of the section and so materially increase the unit stresses. There is so far no satisfactory method devised by which we may accurately determine the centre of pressure, and it is desirable therefore to make allowance for these indeterminable stresses by making domes of ample thickness.

CHAPTER V.

METHODS OF REINFORCEMENT.

THERE are at the present time a great number of systems employing various methods of reinforcement. It is not necessary to describe these in detail, but it may be well to briefly set out the general methods of reinforcement which may be adopted to resist the stresses in reinforced concrete structures.

The reinforcement of *columns and piles* may consist of longitudinal rods tied together by wire ties spaced some distance apart, as shown (Fig. 48). The spacing of the ties for this type of reinforcement should not be more than twenty-four times the diameter of the longitudinal reinforcements for columns, and should be about twelve diameters for piles. At the top of piles the spacing should gradually reduce to about two or three diameters.

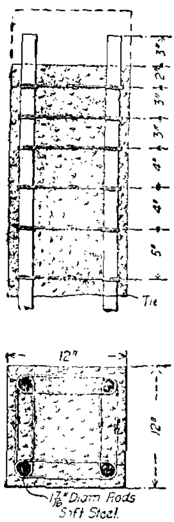


FIG. 48.

Columns and piles are frequently reinforced by spiral bindings or closely spaced hoopings, together with longitudinal rods. If the hooping is to be

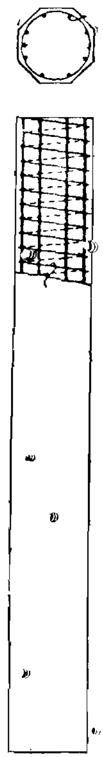


FIG. 49.

considered as adding to the resistance the spacing should not be greater than about one-fifth the diameter hooped

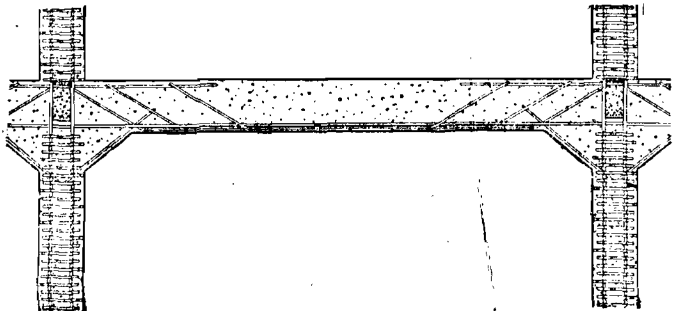


FIG. 50.

core; a very usual spacing is about one-eighth this diameter. This method is shown in Figs. 49 and 50.

The footings of columns or walls are usually reinforced by bars crossing each other parallel to the sides or diagonally or both, and placed near the lower surface. A good method to adopt is to place two series of fairly large sized bars crossing one another at right angles under the base of the column and parallel to the sides, with smaller bars parallel to these from the outer sides of each series to the outer

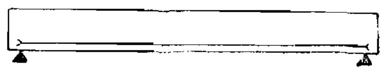


FIG. 51.

FIG. 52.

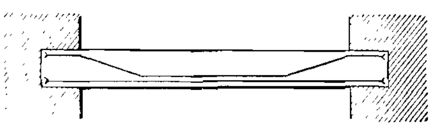


FIG. 53.

edges of the footing. The larger sized bars are calculated to resist the reaction of the four trapezoidal portions of the footing while the smaller bars distribute the reactions from the corners.

When a *beam or slab* is freely supported and the diagonal tensile stresses are resisted by the concrete, it is only necessary to place reinforcements near the under-side, as shown in Fig. 51.

When the member is fixed at the ends or passes over intermediate

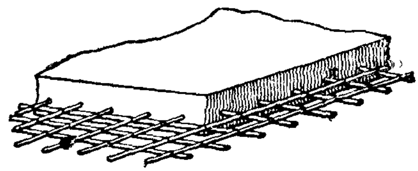


Fig. 54.

supports and the diagonal tensile stresses are resisted by the concrete, it is necessary to place further reinforcements near the top surface over the supports and extending into the pieces, as shown in Fig. 52; or some of the bottom reinforcements may be bent up so as to pass over the supports near the upper surface as shown in Fig. 53.

For *slabs* which are square or

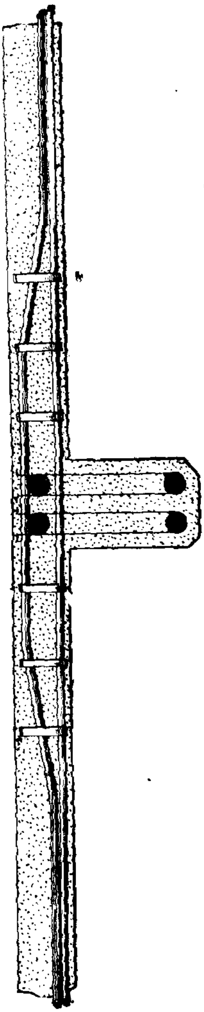


Fig. 55.

when the length is less than twice the width, it will be necessary to provide reinforcements in both directions, as

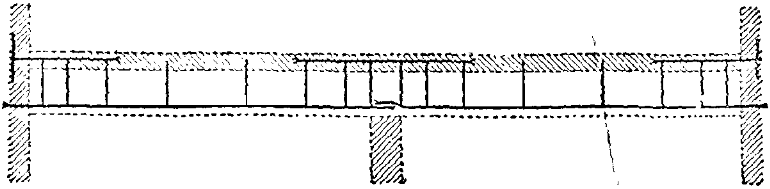


Fig. 56.

shown in Fig. 54, or expanded metal or other mesh reinforcement may be used. It is advisable to provide some

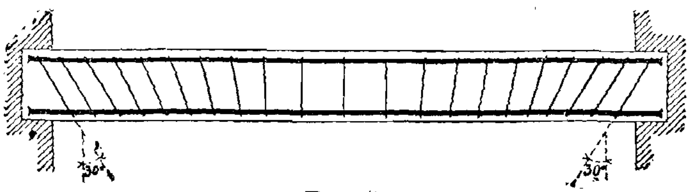


Fig. 57.

transverse reinforcement in all slabs, especially if they may be loaded locally.

If the diagonal tensile stresses are more than can be safely resisted by the concrete alone, it becomes necessary to place reinforcements to resist these. The bent-up bars shown in Fig. 53 provide resistance to diagonal tensile

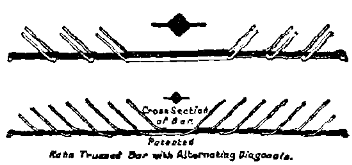


Fig. 58.

stresses, but it is generally necessary to insert further reinforcements for this purpose. Fig. 55 shows a very usual method, the stirrups being flat hoop-iron or round rods. Sometimes the bent-up bars are omitted, and reinforcements are inserted as in Figs. 56 and 57.

It is advisable that such reinforcements should be in some way rigidly attached to the bottom rods, and

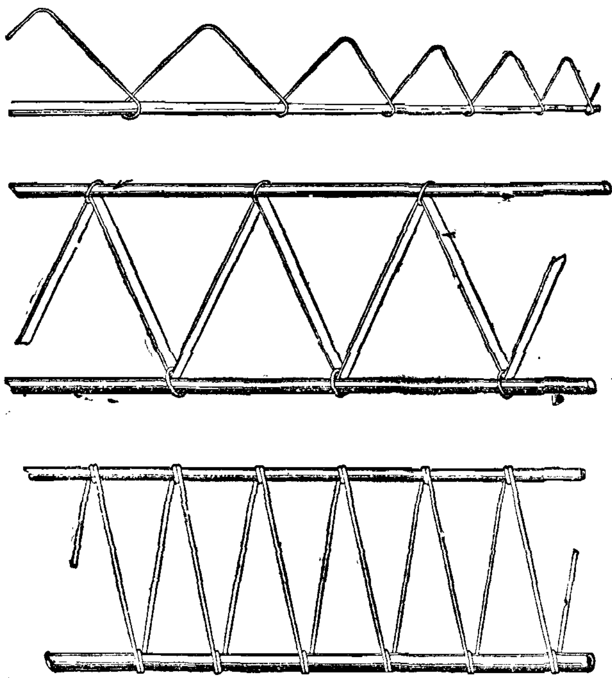


Fig. 59.

they must penetrate nearly to the upper surface, or, better still, be bent over and secured to horizontal bars near this surface. The bottom rods may also be bent up without passing horizontally over the supports as shown in Fig. 50, or the bars may be of a special form, as Fig. 58.

Other methods of reinforcing against diagonal tension are shown in Fig. 59.

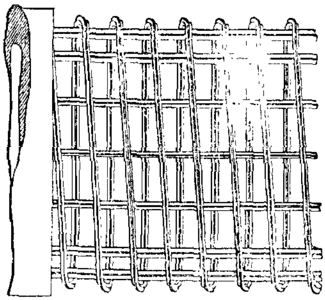


FIG. 60.

If the depth of the beam is insufficient for the concrete to supply adequate resistance to compression, horizontal reinforcements are provided for the entire length near the upper surface, as shown in Figs. 57 and 59.

Cantilevers are reinforced with bars near the upper surface, and these must be firmly secured or embedded in the support. The reinforcement against diagonal tension will be as described for beams.

The tops of columns should be well tied into the beams by continuing the longitudinal bars of the columns well into the beams. The longitudinal bars in the beams must also be carried over the columns and well into the supports.

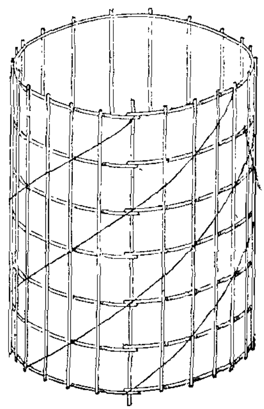


FIG. 61.

The longitudinal bars of columns should be machined flat at their lower ends and bear against a plate inserted in the footings, or be bent out into the footing near the centre of its depth. When there are a series of floors the bars in the columns should be made continuous by being machined at the ends and being connected by sleeves or some other adequate arrangement.

Pipes, reservoirs, and similar structures are reinforced with hoops or spiral windings and longitudinal distribution rods, as shown in Figs. 60 and 61.

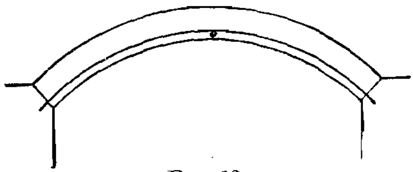


Fig. 62

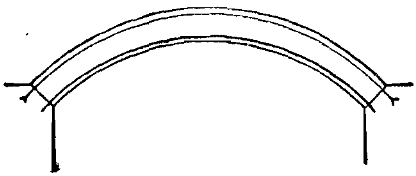


Fig. 63.

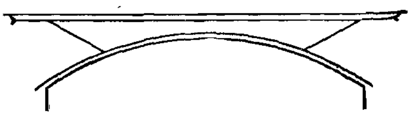


Fig. 64.

Small span *arches* are frequently only reinforced near the intrados, as in Fig. 62, but for large spans it is advisable to insert reinforcements both near the intrados and extrados, as in Fig. 63.

Arches with a flat extrados are very frequently reinforced as shown (Fig. 64), although the central portion of the upper reinforcement is sometimes omitted.

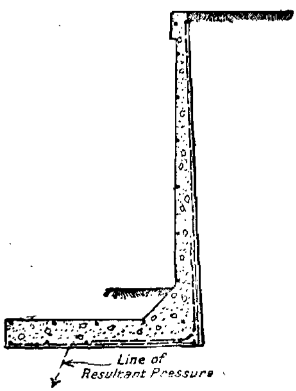


Fig. 65.

In the case of arches with a flat extrados having a small rise in proportion to the span it is necessary to place reinforcements in the vertical planes near the centre to tie the longitudinal bars near the intrados to the concrete, or further bars near the extrados. Generally speaking, it is advisable to use inclined reinforcements in arches to *tie* the upper and lower longitudinal reinforcements together, or to secure the bars near the intrados to the concrete at the extrados. In fact, for all reinforced

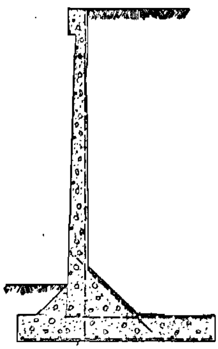


FIG. 66.

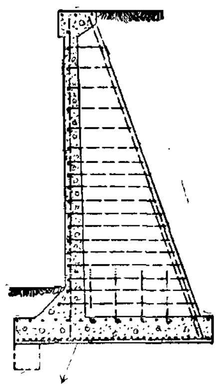


FIG. 67.

concrete work it is well to thoroughly tie the bars to each other and to the concrete.

There are, of course, many structures of a compound nature in which the method of reinforcement depends on the stresses to be resisted by the component parts, but the general description given should be a sufficient guide to the rational reinforcement of such structures.

Figs. 65, 66, and 67 show three methods of designing retaining walls.

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